Question 1:

Evaluate the Given limit: $\lim\limits_{x \to 3} x+3$

Answer:

$\lim\limits_{x \to 3} x+3 = 3+3 = 6$

Question 2:

Evaluate the Given limit: $\lim\limits_{x \to \pi}\left(x-\frac{22}{7}\right)$

Answer:

$\lim\limits_{x \to \pi}\left(x-\frac{22}{7}\right) = \left(\pi-\frac{22}{7}\right)$

Question 3:

Evaluate the Given limit: $\lim\limits_{r \to 1} \pi r^2$

Answer:

$\lim\limits_{r \to 1} \pi r^2 = \pi (1)^2 = \pi$

Question 4:

Evaluate the Given limit: $\lim\limits_{x \to 4} \frac{4x+3}{x-2}$

Answer:

$\lim\limits_{x \to 4} \frac{4x+3}{x-2} = \frac{4(4)+3}{4-2} = \frac{16+3}{2} = \frac{19}{2}$

Question 5:

Evaluate the Given limit: $\lim\limits_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$

Answer:

$$\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1} = \frac{(-1)^{10}+(-1)^5+1}{-1-1} = \frac{1-1+1}{-2} = -\frac{1}{2}$$

Question 6:

Evaluate the Given limit: $\lim_{x \to 0} \frac{(x+1)^5-1}{x}$

Answer:

$$\lim_{x \to 0} \frac{(x+1)^5-1}{x}$$

Put x + 1 = y so that y → 1 as x → 0.

$$\text{Accordingly, } \lim_{x \to 0} \frac{(x+1)^5-1}{x} = \lim_{y \to 1} \frac{y^5-1}{y-1}$$

$$= \lim_{y \to 1} \frac{y^5-1^5}{y-1}$$

$$= 5.1^{5-1} \qquad \left[\lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 5$$

$$\therefore \lim_{x \to 0} \frac{(x+5)^5-1}{x} = 5$$

Question 7:

Evaluate the Given limit: $\lim_{x \to 2} \frac{3x^2-x-10}{x^2-4}$

Answer:

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$$
$$=\lim_{x\to 2}\frac{3x+5}{x+2}$$
$$=\frac{3(2)+5}{2+2}$$
$$=\frac{11}{4}$$

Question 8:

Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Answer:

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$
$$=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{2x+1}$$
$$=\frac{(3+3)(3^2+9)}{2(3)+1}$$
$$=\frac{6\times 18}{7}$$
$$=\frac{108}{7}$$

Question 9:

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+b}{cx+1}$

Answer:

$$\lim_{x\to 0}\frac{ax+b}{cx+1}=\frac{a(0)+b}{c(0)+1}=b$$

Question 10:

Evaluate the Given limit: $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$

Answer:

$$\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$$

At z = 1, the value of the given function takes the form $\frac{0}{0}$.

Put $z^{\frac{1}{6}} = x$ so that z →1 as x → 1.

$$\text{Accordingly, } \lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1}$$

$$= 2.1^{2-1} \qquad \left[\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= 2$$

$$\therefore \lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = 2$$

Question 11:

Evaluate the Given limit: $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$

Answer:

$$\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a} = \frac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a}$$

$$= \frac{a + b + c}{a + b + c}$$

$$= 1 \qquad [a + b + c \neq 0]$$

Question 12:

Evaluate the Given limit: $\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Answer:

$$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$$

At x = –2, the value of the given function takes the form $\frac{0}{0}$.

$$\begin{aligned}
\text{Now, } \lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} &= \lim_{x \to -2} \frac{\left(\frac{2+x}{2x}\right)}{x+2} \\
&= \lim_{x \to -2} \frac{1}{2x} \\
&= \frac{1}{2(-2)} = \frac{-1}{4}
\end{aligned}$$

Question 13:

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{bx}$

Answer:

$$\lim_{x \to 0} \frac{\sin ax}{bx}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

$$\begin{aligned}
\text{Now, } \lim_{x \to 0} \frac{\sin ax}{bx} &= \lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{ax}{bx} \\
&= \lim_{x \to 0} \left(\frac{\sin ax}{ax}\right) \times \left(\frac{a}{b}\right) \\
&= \frac{a}{b} \lim_{ax \to 0} \left(\frac{\sin ax}{ax}\right) && [x \to 0 \Rightarrow ax \to 0] \\
&= \frac{a}{b} \times 1 && \left[\lim_{y \to 0} \frac{\sin y}{y} = 1\right] \\
&= \frac{a}{b}
\end{aligned}$$

Question 14:

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{\sin bx}, \ a, \ b \neq 0$

Answer:

$$\lim_{x \to 0} \frac{\sin ax}{\sin bx}, \ a, \ b \neq 0$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x \to 0} \frac{\sin ax}{\sin bx} = \lim_{x \to 0} \frac{\left(\frac{\sin ax}{ax}\right) \times ax}{\left(\frac{\sin bx}{bx}\right) \times bx}$$

$$= \left(\frac{a}{b}\right) \times \frac{\lim_{ax \to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{bx \to 0}\left(\frac{\sin bx}{bx}\right)} \qquad \left[\begin{array}{l} x \to 0 \Rightarrow ax \to 0 \\ \text{and } x \to 0 \Rightarrow bx \to 0 \end{array}\right]$$

$$= \left(\frac{a}{b}\right) \times \frac{1}{1} \qquad \left[\lim_{y \to 0} \frac{\sin y}{y} = 1\right]$$

$$= \frac{a}{b}$$

Page No 302:

Question 15:

Evaluate the Given limit: $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

Answer:

$$\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

It is seen that x → π ⇒ (π – x) → 0

$$\therefore \lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \frac{1}{\pi} \lim_{(\pi - x) \to 0} \frac{\sin(\pi - x)}{(\pi - x)}$$

$$= \frac{1}{\pi} \times 1 \qquad \left[\lim_{y \to 0} \frac{\sin y}{y} = 1\right]$$

$$= \frac{1}{\pi}$$

Question 16:

Evaluate the given limit: $\lim_{x \to 0} \frac{\cos x}{\pi - x}$

Answer:

$$\lim_{x \to 0} \frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$$

Question 17:

Evaluate the Given limit: $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

Answer:

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1} \qquad \left[\cos x=1-2\sin^2\frac{x}{2}\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$=4\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$=4\frac{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} \qquad \left[x\to 0\Rightarrow\frac{x}{2}\to 0\right]$$

$$=4\frac{1^2}{1^2} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=4$$

Question 18:

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Answer:

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$$

$$=\frac{1}{b}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times\frac{1}{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times(a+\cos 0)\qquad\left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{a+1}{b}$$

Question 19:

Evaluate the Given limit: $\lim_{x\to 0} x\sec x$

Answer:

$$\lim_{x\to 0} x\sec x=\lim_{x\to 0}\frac{x}{\cos x}=\frac{0}{\cos 0}=\frac{0}{1}=0$$

Question 20:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}\ \ a,b,a+b\neq 0$

Answer:

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax+bx}{ax+bx\left(\frac{\sin bx}{bx}\right)}$$

$$=\frac{\left(\lim\limits_{ax\to 0}\frac{\sin ax}{ax}\right)\times\lim\limits_{x\to 0}(ax)+\lim\limits_{x\to 0}bx}{\lim\limits_{x\to 0}ax+\lim\limits_{x\to 0}bx\left(\lim\limits_{bx\to 0}\frac{\sin bx}{bx}\right)} \qquad [\text{As } x\to 0\Rightarrow ax\to 0 \text{ and } bx\to 0]$$

$$=\frac{\lim\limits_{x\to 0}(ax)+\lim\limits_{x\to 0}bx}{\lim\limits_{x\to 0}ax+\lim\limits_{x\to 0}bx} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{\lim\limits_{x\to 0}(ax+bx)}{\lim\limits_{x\to 0}(ax+bx)}$$

$$=\lim_{x\to 0}(1)$$

$$=1$$

Question 21:

Evaluate the Given limit: $\lim\limits_{x\to 0}(\operatorname{cosec} x-\cot x)$

Answer:

At x = 0, the value of the given function takes the form $\infty-\infty$.

Now,

$$\lim_{x\to 0}(\operatorname{cosec} x-\cot x)$$

$$=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

$$=\frac{\lim\limits_{x\to 0}\frac{1-\cos x}{x}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}$$

$$=\frac{0}{1} \qquad \left[\lim_{x\to 0}\frac{1-\cos x}{x}=0 \text{ and } \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=0$$

Question 22:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

Answer:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

At $x=\frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x-\frac{\pi}{2}=y$ so that $x\to\frac{\pi}{2},\ y\to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0} \frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$= \lim_{y\to 0} \frac{\tan(\pi+2y)}{y}$$

$$= \lim_{y\to 0} \frac{\tan 2y}{y} \qquad \left[\tan(\pi+2y) = \tan 2y\right]$$

$$= \lim_{y\to 0} \frac{\sin 2y}{y\cos 2y}$$

$$= \lim_{y\to 0}\left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y}\right)$$

$$= \left(\lim_{2y\to 0} \frac{\sin 2y}{2y}\right) \times \lim_{y\to 0}\left(\frac{2}{\cos 2y}\right) \qquad \left[y\to 0 \Rightarrow 2y\to 0\right]$$

$$= 1\times\frac{2}{\cos 0} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 1\times\frac{2}{1}$$

$$= 2$$

Question 23:

Find $\lim\limits_{x\to 0}$ f(x) and $\lim\limits_{x\to 1}$ f(x), where f(x) = $\begin{cases} 2x+3, & x\le 0 \\ 3(x+1), & x>0 \end{cases}$

Answer:

The given function is

f(x) = $\begin{cases} 2x+3, & x\le 0 \\ 3(x+1), & x>0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left[2x+3\right] = 2(0)+3 = 3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3(x+1) = 3(0+1) = 3$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 3$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 6$$

Question 24:

Find $\lim_{x \to 1}$ f(x), where f(x) = $\begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$

Answer:

The given function is

$$f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} \left[x^2 - 1\right] = 1^2 - 1 = 1 - 1 = 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} \left[-x^2 - 1\right] = -1^2 - 1 = -1 - 1 = -2$$

It is observed that $\lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$.

Hence, $\lim_{x \to 1} f(x)$ does not exist.

Question 25:

Evaluate $\lim_{x \to 0}$ f(x), where f(x) = $\begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Answer:

The given function is

f(x) = $\begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{|x|}{x}\right]$$

$$= \lim_{x\to 0}\left(\frac{-x}{x}\right) \qquad [\text{When } x \text{ is negaitve, } |x| = -x]$$

$$= \lim_{x\to 0}(-1)$$

$$= -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{|x|}{x}\right]$$

$$= \lim_{x\to 0}\left[\frac{x}{x}\right] \qquad [\text{When } x \text{ is positive, } |x| = x]$$

$$= \lim_{x\to 0}(1)$$

$$= 1$$

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 26:

Find $\lim_{x\to 0}$ f(x), where f(x) = $\begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Answer:

The given function is

$$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left[\frac{x}{|x|}\right]\\ &= \lim_{x\to 0}\left[\frac{x}{-x}\right] && [\text{When } x<0,\ |x|=-x]\\ &= \lim_{x\to 0}(-1)\\ &= -1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left[\frac{x}{|x|}\right]\\ &= \lim_{x\to 0}\left[\frac{x}{x}\right] && [\text{When } x>0,\ |x|=x]\\ &= \lim_{x\to 0}(1)\\ &= 1\end{aligned}$$

It is observed that $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 27:

Find $\lim_{x\to 5}$ f(x), where f(x) = $|x|-5$

Answer:

The given function is f(x) = $|x|-5$.

$$\begin{aligned}\lim_{x\to 5^-} f(x) &= \lim_{x\to 5^-}\left[|x|-5\right]\\ &= \lim_{x\to 5}(x-5) && [\text{When } x>0,\ |x|=x]\\ &= 5-5\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 5^+} f(x) &= \lim_{x\to 5^+}\left(|x|-5\right)\\ &= \lim_{x\to 5}(x-5) && [\text{When } x>0,\ |x|=x]\\ &= 5-5\\ &= 0\end{aligned}$$

$\therefore\ \lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=0$

Hence, $\lim_{x\to 5} f(x)=0$

Question 28:

Suppose f(x) = $\begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$ and if $\lim_{x\to 1}$ f(x) = f(1) what are possible values of a and b?

Answer:

The given function is

$$f(x)=\begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$$

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}(a+bx)=a+b$

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(b-ax)=b-a$

$f(1)=4$

It is given that $\lim_{x\to 1} f(x)=f(1)$.

$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x)=f(1)$

$\Rightarrow a+b=4$ and $b-a=4$

On solving these two equations, we obtain $a=0$ and $b=4$.

Thus, the respective possible values of a and b are 0 and 4.

Page No 303:

Question 29:

Let $a_1, a_2, \ldots, a_n$ be fixed real numbers and define a function

$f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$

What is $\lim_{x\to a_1}$ f(x)? For some $a \neq a_1, a_2, \ldots, a_n$, compute $\lim_{x\to a}$ f(x).

Answer:

The given function is $f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$.

$$\lim_{x\to a_1} f(x) = \lim_{x\to a_1}\left[(x-a_1)(x-a_2)\ldots(x-a_n)\right]$$
$$= \left[\lim_{x\to a_1}(x-a_1)\right]\left[\lim_{x\to a_1}(x-a_2)\right]\ldots\left[\lim_{x\to a_1}(x-a_n)\right]$$
$$= (a_1-a_1)(a_1-a_2)\ldots(a_1-a_n) = 0$$
$$\therefore \lim_{x\to a_1} f(x) = 0$$

Now, $\lim_{x\to a} f(x) = \lim_{x\to a}\left[(x-a_1)(x-a_2)\ldots(x-a_n)\right]$
$$= \left[\lim_{x\to a}(x-a_1)\right]\left[\lim_{x\to a}(x-a_2)\right]\ldots\left[\lim_{x\to a}(x-a_n)\right]$$
$$= (a-a_1)(a-a_2)\ldots\ldots(a-a_n)$$
$$\therefore \lim_{x\to a} f(x) = (a-a_1)(a-a_2)\ldots(a-a_n)$$

Question 30:

If f(x) = $\begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$.

For what value (s) of a does $\lim_{x\to a}$ f(x) exists?

Answer:

The given function is

$$f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$$

When $a = 0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(|x|+1)$$
$$= \lim_{x\to 0}(-x+1) \qquad [\text{If } x<0,\ |x| = -x]$$
$$= -0+1$$
$$= 1$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(|x|-1)$$
$$= \lim_{x\to 0}(x-1) \qquad [\text{If } x>0,\ |x| = x]$$
$$= 0-1$$
$$= -1$$

Here, it is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When a < 0,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} (|x|+1)$$
$$= \lim_{x\to a} (-x+1) \qquad [x < a < 0 \Rightarrow |x| = -x]$$
$$= -a+1$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} (|x|+1)$$
$$= \lim_{x\to a} (-x+1) \qquad [a < x < 0 \Rightarrow |x| = -x]$$
$$= -a+1$$

$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = -a+1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a < 0$.

When a > 0

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} (|x|-1)$$
$$= \lim_{x\to a} (x-1) \qquad [0 < x < a \Rightarrow |x| = x]$$
$$= a-1$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} (|x|-1)$$
$$= \lim_{x\to a} (x-1) \qquad [0 < a < x \Rightarrow |x| = x]$$
$$= a-1$$

$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = a-1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim_{x\to a} f(x)$ exists for all a ≠ 0.

Question 31:

If the function f(x) satisfies $\lim_{x\to 1} \frac{f(x)-2}{x^2-1} = \pi$, evaluate $\lim_{x\to 1} f(x)$.

Answer:

$$\lim_{x\to1}\frac{f(x)-2}{x^2-1}=\pi$$

$$\Rightarrow\frac{\lim\limits_{x\to1}(f(x)-2)}{\lim\limits_{x\to1}(x^2-1)}=\pi$$

$$\Rightarrow\lim_{x\to1}(f(x)-2)=\pi\lim_{x\to1}(x^2-1)$$

$$\Rightarrow\lim_{x\to1}(f(x)-2)=\pi(1^2-1)$$

$$\Rightarrow\lim_{x\to1}(f(x)-2)=0$$

$$\Rightarrow\lim_{x\to1}f(x)-\lim_{x\to1}2=0$$

$$\Rightarrow\lim_{x\to1}f(x)-2=0$$

$$\therefore\lim_{x\to1}f(x)=2$$

Question 32:

If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$. For what integers m and n does $\lim\limits_{x\to0}f(x)$ and $\lim\limits_{x\to1}f(x)$ exist?

Answer:

The given function is

$$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

$$\begin{aligned}\lim_{x\to0^-}f(x)&=\lim_{x\to0}(mx^2+n)\\&=m(0)^2+n\\&=n\end{aligned}$$

$$\begin{aligned}\lim_{x\to0^+}f(x)&=\lim_{x\to0}(nx+m)\\&=n(0)+m\\&=m.\end{aligned}$$

Thus, $\lim\limits_{x\to0}f(x)$ exists if m = n.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(nx+m)$$
$$= n(1)+m$$
$$= m+n$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(nx^3+m)$$
$$= n(1)^3+m$$
$$= m+n$$
$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x).$$

Thus, $\lim_{x\to 1} f(x)$ exists for any integral value of m and n.

Page No 312:

Question 1:

Find the derivative of $x^2 - 2$ at x = 10.

Answer:

Let f(x) = $x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$
$$= \lim_{h\to 0}\frac{\left[(10+h)^2-2\right]-(10^2-2)}{h}$$
$$= \lim_{h\to 0}\frac{10^2+2.10.h+h^2-2-10^2+2}{h}$$
$$= \lim_{h\to 0}\frac{20h+h^2}{h}$$
$$= \lim_{h\to 0}(20+h) = (20+0) = 20$$

Thus, the derivative of $x^2 - 2$ at x = 10 is 20.

Question 2:

Find the derivative of 99x at x = 100.

Answer:

Let f(x) = 99x. Accordingly,

$$\begin{aligned}f'(100) &= \lim_{h\to 0}\frac{f(100+h)-f(100)}{h}\\ &= \lim_{h\to 0}\frac{99(100+h)-99(100)}{h}\\ &= \lim_{h\to 0}\frac{99\times 100+99h-99\times 100}{h}\\ &= \lim_{h\to 0}\frac{99h}{h}\\ &= \lim_{h\to 0}(99)=99\end{aligned}$$

Thus, the derivative of 99x at x = 100 is 99.

Question 3:

Find the derivative of x at x = 1.

Answer:

Let f(x) = x. Accordingly,

$$\begin{aligned}f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\ &= \lim_{h\to 0}\frac{(1+h)-1}{h}\\ &= \lim_{h\to 0}\frac{h}{h}\\ &= \lim_{h\to 0}(1)\\ &= 1\end{aligned}$$

Thus, the derivative of x at x = 1 is 1.

Question 4:

Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$

(ii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

Answer:

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\left[(x+h)^3-27\right]-\left(x^3-27\right)}{h}\\
&= \lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}\\
&= \lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h}\\
&= \lim_{h\to 0}\left(h^2+3x^2+3xh\right)\\
&= 0+3x^2+0=3x^2
\end{aligned}
$$

(ii) Let f(x) = (x – 1) (x – 2). Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}\\
&= \lim_{h\to 0}\frac{\left(x^2+hx-2x+hx+h^2-2h-x-h+2\right)-\left(x^2-2x-x+2\right)}{h}\\
&= \lim_{h\to 0}\frac{\left(hx+hx+h^2-2h-h\right)}{h}\\
&= \lim_{h\to 0}\frac{2hx+h^2-3h}{h}\\
&= \lim_{h\to 0}(2x+h-3)\\
&= (2x+0-3)\\
&= 2x-3
\end{aligned}
$$

(iii) Let $f(x)=\frac{1}{x^2}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$$
$$=\lim_{h\to 0}\left[\frac{-h-2x}{x^2(x+h)^2}\right]$$
$$=\frac{0-2x}{x^2(x+0)^2}=\frac{-2}{x^3}$$

(iv) Let $f(x)=\frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$
$$=\frac{-2}{(x-1)(x-1)}=\frac{-2}{(x-1)^2}$$

Question 5:

For the function

$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1$

Prove that $f'(1)=100f'(0)$

Answer:

The given function is

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1\right]$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+...+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+...+\frac{2x}{2}+1+0$$

$$=x^{99}+x^{98}+...+x+1$$

$$\therefore f'(x)=x^{99}+x^{98}+...+x+1$$

At $x=0$,

$$f'(0)=1$$

At $x=1$,

$$f'(1)=1^{99}+1^{98}+...+1+1=\left[1+1+...+1+1\right]_{100\text{ terms}}=1\times 100=100$$

Thus, $f'(1)=100\times f^1(0)$

Page No 313:

Question 6:

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$ for some fixed real number a.

Answer:

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$

$$\therefore f'(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n\right)$$
$$=\frac{d}{dx}\left(x^n\right)+a\frac{d}{dx}\left(x^{n-1}\right)+a^2\frac{d}{dx}\left(x^{n-2}\right)+...+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}+a^n(0)$$
$$=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}$$

Question 7:

For some constants a and b, find the derivative of

(i) (x – a) (x – b) (ii) $(ax^2 + b)^2$ (iii) $\frac{x-a}{x-b}$

Answer:

(i) Let f (x) = (x – a) (x – b)

$$\Rightarrow f(x)=x^2-(a+b)x+ab$$
$$\therefore f'(x)=\frac{d}{dx}\left(x^2-(a+b)x+ab\right)$$
$$=\frac{d}{dx}\left(x^2\right)-(a+b)\frac{d}{dx}(x)+\frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$, we obtain

$$f'(x)=2x-(a+b)+0=2x-a-b$$

(ii) Let $f(x)=\left(ax^2+b\right)^2$

$$\Rightarrow f(x)=a^2x^4+2abx^2+b^2$$
$$\therefore f'(x)=\frac{d}{dx}\left(a^2x^4+2abx^2+b^2\right)=a^2\frac{d}{dx}\left(x^4\right)+2ab\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(b^2\right)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$$f'(x)=a^2\left(4x^3\right)+2ab(2x)+b^2(0)$$
$$=4a^2x^3+4abx$$
$$=4ax\left(ax^2+b\right)$$

(iii) Let $f(x)=\frac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$
$$=\frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$
$$=\frac{x-b-x+a}{(x-b)^2}$$
$$=\frac{a-b}{(x-b)^2}$$

Question 8:

Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant a.

Answer:

$$\text{Let } f(x)=\frac{x^n-a^n}{x-a}$$
$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$
$$=\frac{(x-a)(nx^{n-1}-0)-(x^n-a^n)}{(x-a)^2}$$
$$=\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

Question 9:

Find the derivative of

(i) $2x-\frac{3}{4}$ (ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$ (iv) $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$ (vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Answer:

(i) Let $f(x) = 2x - \frac{3}{4}$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x - \frac{3}{4}\right) \\ &= 2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

(ii) Let f (x) = $(5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (5x^3 + 3x - 1)\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}(5x^3 + 3x - 1) \\ &= (5x^3 + 3x - 1)(1) + (x-1)(5.3x^2 + 3 - 0) \\ &= (5x^3 + 3x - 1) + (x-1)(15x^2 + 3) \\ &= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3 \\ &= 20x^3 - 15x^2 + 6x - 4 \end{aligned}$$

(iii) Let f (x) = $x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= x^{-3}\frac{d}{dx}(5+3x) + (5+3x)\frac{d}{dx}(x^{-3}) \\ &= x^{-3}(0+3) + (5+3x)(-3x^{-3-1}) \\ &= x^{-3}(3) + (5+3x)(-3x^{-4}) \\ &= 3x^{-3} - 15x^{-4} - 9x^{-3} \\ &= -6x^{-3} - 15x^{-4} \\ &= -3x^{-3}\left(2 + \frac{5}{x}\right) \\ &= \frac{-3x^{-3}}{x}(2x+5) \\ &= \frac{-3}{x^4}(5+2x) \end{aligned}$$

(iv) Let f (x) = $x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$f'(x) = x^5 \frac{d}{dx}(3-6x^{-9}) + (3-6x^{-9})\frac{d}{dx}(x^5)$$
$$= x^5\{0-6(-9)x^{-9-1}\} + (3-6x^{-9})(5x^4)$$
$$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$$
$$= 54x^{-5} + 15x^4 - 30x^{-5}$$
$$= 24x^{-5} + 15x^4$$
$$= 15x^4 + \frac{24}{x^5}$$

(v) Let f (x) = x^{-4} (3 – $4x^{-5}$)

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}(3-4x^{-5}) + (3-4x^{-5})\frac{d}{dx}(x^{-4})$$
$$= x^{-4}\{0-4(-5)x^{-5-1}\} + (3-4x^{-5})(-4)x^{-4-1}$$
$$= x^{-4}(20x^{-6}) + (3-4x^{-5})(-4x^{-5})$$
$$= 20x^{-10} - 12x^{-5} + 16x^{-10}$$
$$= 36x^{-10} - 12x^{-5}$$
$$= -\frac{12}{x^5} + \frac{36}{x^{10}}$$

(vi) Let f (x) = $\frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$
$$= \left[\frac{(x+1)(0)-2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x)-(x^2)(3)}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2-2x^2}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

Question 10:

Find the derivative of cos x from first principle.

Answer:

Let f (x) = cos x. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\
&= \lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]\\
&= \lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]\\
&= \lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]\\
&= -\cos x\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\
&= -\cos x(0)-\sin x(1) \qquad \left[\lim_{h\to 0}\frac{1-\cos h}{h}=0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h}=1\right]\\
&= -\sin x\\
\therefore f'(x) &= -\sin x
\end{aligned}
$$

Question 11:

Find the derivative of the following functions:

(i) sin x cos x (ii) sec x (iii) 5 sec x + 4 cos x

(iv) cosec x (v) 3cot x + 5cosec x

(vi) 5sin x – 6cos x + 7 (vii) 2tan x – 7sec x

Answer:

(i) Let f (x) = sin x cos x. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)\cos(x+h)-\sin x\cos x}{h} \\
&= \lim_{h\to 0}\frac{1}{2h}\left[2\sin(x+h)\cos(x+h)-2\sin x\cos x\right] \\
&= \lim_{h\to 0}\frac{1}{2h}\left[\sin 2(x+h)-\sin 2x\right] \\
&= \lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}\cdot\sin\frac{2x+2h-2x}{2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos(2x+h)\sin h\right] \\
&= \lim_{h\to 0}\cos(2x+h).\lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos(2x+0).1 \\
&= \cos 2x
\end{aligned}
$$

(ii) Let f (x) = sec x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.1.\frac{\sin x}{\cos x}$$
$$=\sec x\tan x$$

(iii) Let f (x) = 5 sec x + 4 cos x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-[5\sec x+4\cos x]}{h}$$
$$=5\lim_{h\to 0}\frac{[\sec(x+h)-\sec x]}{h}+4\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$
$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos(x+h)-\cos x]$$
$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos x\cos h-\sin x\sin h-\cos x]$$
$$=\frac{5}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[-\cos x(1-\cos h)-\sin x\sin h]$$
$$=\frac{5}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]+4\left[-\cos x\lim_{h\to 0}\frac{(1-\cos h)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$
$$=\frac{5}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right]+4[(-\cos x).(0)-(\sin x).1]$$
$$=\frac{5}{\cos x}.\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]-4\sin x$$
$$=\frac{5}{\cos x}.\frac{\sin x}{\cos x}.1-4\sin x$$
$$=5\sec x\tan x.-4\sin x$$

(iv) Let f (x) = cosec x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x$$

(v) Let f (x) = 3cot x + 5cosec x. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3\cot(x+h) + 5\text{cosec}(x+h) - 3\cot x - 5\text{cosec}\, x}{h}$$

$$= 3\lim_{h \to 0} \frac{1}{h}\left[\cot(x+h) - \cot x\right] + 5\lim_{h \to 0} \frac{1}{h}\left[\text{cosec}(x+h) - \text{cosec}\, x\right] \qquad ...(1)$$

Now, $\lim_{h \to 0} \frac{1}{h}\left[\cot(x+h) - \cot x\right]$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cos(x+h)\sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x - x - h)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(-h)}{\sin x \sin(x+h)}\right]$$

$$= -\left(\lim_{h \to 0} \frac{\sin h}{h}\right) \cdot \left(\lim_{h \to 0} \frac{1}{\sin x \cdot \sin(x+h)}\right)$$

$$= -1 . \frac{1}{\sin x \cdot \sin(x+0)} = \frac{-1}{\sin^2 x} = -\text{cosec}^2 x \qquad ...(2)$$

$$\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$f'(x)=-3\operatorname{cosec}^2 x-5\operatorname{cosec}x\cot x$$

(vi) Let f (x) = 5sin x – 6cos x + 7. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[5\{\sin(x+h)-\sin x\}-6\{\cos(x+h)-\cos x\}\right]$$
$$=5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h)-\sin x\right]-6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$
$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$
$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$
$$=5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$
$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]-6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$
$$=5\cos x.1-6\left[(-\cos x).(0)-\sin x.1\right]$$
$$=5\cos x+6\sin x$$

(vii) Let f (x) = 2 tan x – 7 sec x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\left\{\tan(x+h)-\tan x\right\}-7\left\{\sec(x+h)-\sec x\right\}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$=2.1.\frac{1}{\cos x\cos x}-7.1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$=2\sec^2 x-7\sec x\tan x$$

Page No 317:

Question 1:

Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin(x+1)$

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Answer:

(i) Let $f(x) = -x$. Accordingly, $f(x+h)=-(x+h)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{-(x+h)-(-x)}{h} \\
&= \lim_{h\to 0}\frac{-x-h+x}{h} \\
&= \lim_{h\to 0}\frac{-h}{h} \\
&= \lim_{h\to 0}(-1) = -1
\end{aligned}
$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}-\left(\frac{-1}{x}\right)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}+\frac{1}{x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{h}{x(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{x(x+h)} \\
&= \frac{1}{x\cdot x} = \frac{1}{x^2}
\end{aligned}
$$

(iii) Let f(x) = sin (x + 1). Accordingly, $f(x+h) = \sin(x+h+1)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+1)-\sin(x+1)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right] \\
&= \lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right] \\
&= \lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right] \\
&= \cos\left(\frac{2x+0+2}{2}\right)\cdot 1 \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right] \\
&= \cos(x+1)
\end{aligned}
$$

(iv) Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$. Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]
\end{aligned}
$$

$$= \lim_{h\to 0} \frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

$$= \lim_{h\to 0} \left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$= \lim_{h\to 0} \left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= -\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right).1$$

$$= -\sin\left(x-\frac{\pi}{8}\right)$$

Question 2:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): (x + a)

Answer:

Let f(x) = x + a. Accordingly, $f(x+h) = x+h+a$

By first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{x+h+a-x-a}{h}$$
$$= \lim_{h\to 0}\left(\frac{h}{h}\right)$$
$$= \lim_{h\to 0}(1)$$
$$= 1$$

Question 3:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $(px+q)\left(\frac{r}{x}+s\right)$

Answer:

Let $f(x)=(px+q)\left(\frac{r}{x}+s\right)$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (px+q)\left(\frac{r}{x}+s\right)' + \left(\frac{r}{x}+s\right)(px+q)' \\ &= (px+q)\left(rx^{-1}+s\right)' + \left(\frac{r}{x}+s\right)(p) \\ &= (px+q)\left(-rx^{-2}\right) + \left(\frac{r}{x}+s\right)p \\ &= (px+q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x}+s\right)p \\ &= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps \\ &= ps - \frac{qr}{x^2} \end{aligned}$$

Question 4:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): (ax + b) (cx + d)^2

Answer:

Let $f(x)=(ax+b)(cx+d)^2$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (ax+b)\frac{d}{dx}(cx+d)^2 + (cx+d)^2\frac{d}{dx}(ax+b) \\ &= (ax+b)\frac{d}{dx}\left(c^2x^2+2cdx+d^2\right) + (cx+d)^2\frac{d}{dx}(ax+b) \\ &= (ax+b)\left[\frac{d}{dx}\left(c^2x^2\right)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right] + (cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right] \\ &= (ax+b)\left(2c^2x+2cd\right) + \left(cx+d^2\right)a \\ &= 2c(ax+b)(cx+d) + a(cx+d)^2 \end{aligned}$$

Question 5:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{ax+b}{cx+d}$

Answer:

Let $f(x)=\frac{ax+b}{cx+d}$

By quotient rule,

$$f'(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$=\frac{ad-bc}{(cx+d)^2}$$

Question 6:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers):

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

Answer:

$$\text{Let } f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1}, \text{ where } x\neq 0$$

By quotient rule,

$$f'(x)=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{x-1-x-1}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{-2}{(x-1)^2},\ x\neq 0,\ 1$$

Question 7:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{1}{ax^2+bx+c}$

Answer:

Let $f(x)=\frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x)=\frac{(ax^2+bx+c)\frac{d}{dx}(1)-\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

$$=\frac{(ax^2+bx+c)(0)-(2ax+b)}{(ax^2+bx+c)^2}$$

$$=\frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Question 8:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{ax+b}{px^2+qx+r}$

Answer:

Let $f(x)=\frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f'(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$
$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$
$$=\frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{\left(px^2+qx+r\right)^2}$$
$$=\frac{-apx^2-2bpx+ar-bq}{\left(px^2+qx+r\right)^2}$$

Question 9:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{px^2+qx+r}{ax+b}$

Answer:

Let $f(x)=\frac{px^2+qx+r}{ax+b}$

By quotient rule,

$$f'(x)=\frac{(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)-\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$
$$=\frac{(ax+b)(2px+q)-\left(px^2+qx+r\right)(a)}{(ax+b)^2}$$
$$=\frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$
$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 10:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

Answer:

$$\text{Let } f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x) \qquad \left[\frac{d}{dx}\left(x^n\right)=nx^{n-1}\text{and}\frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

Question 11:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $4\sqrt{x}-2$

Answer:

$$\text{Let } f(x)=4\sqrt{x}-2$$

$$f'(x)=\frac{d}{dx}\left(4\sqrt{x}-2\right)=\frac{d}{dx}\left(4\sqrt{x}\right)-\frac{d}{dx}(2)$$

$$=4\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-0=4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$=\left(2x^{-\frac{1}{2}}\right)=\frac{2}{\sqrt{x}}$$

Question 12:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $(ax + b)^n$

Answer:

$$\text{Let } f(x)=(ax+b)^n\text{. Accordingly, } f(x+h)=\{a(x+h)+b\}^n=(ax+ah+b)^n$$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h}$$

$$=\lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{\left(1+\frac{ah}{ax+b}\right)^n-1}{h}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{n}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{\underline{|2}}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right]$$

(Using binomial theorem)

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{\underline{|2}(ax+b)^2}+\ldots(\text{Terms containing higher degrees of } h)\right]$$

$$=(ax+b)^n\lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h}{\underline{|2}(ax+b)^2}+\ldots\right]$$

$$=(ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$=na\frac{(ax+b)^n}{(ax+b)}$$

$$=na(ax+b)^{n-1}$$

Question 13:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $(ax + b)^n (cx + d)^m$

Answer:

Let $f(x)=(ax+b)^n(cx+d)^m$

By Leibnitz product rule,

$$f'(x)=(ax+b)^n\frac{d}{dx}(cx+d)^m+(cx+d)^m\frac{d}{dx}(ax+b)^n \qquad \ldots(1)$$

Now, let $f_1(x)=(cx+d)^m$

$$f_1(x+h)=(cx+ch+d)^m$$

$$f_1'(x) = \lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$
$$= \lim_{h\to 0}\frac{(cx+ch+d)^m-(cx+d)^m}{h}$$
$$=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]$$
$$=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{(c^2h^2)}{(cx+d)^2}+\ldots\right)-1\right]$$
$$=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+\ldots(\text{Terms containing higher degrees of } h)\right]$$
$$=(cx+d)^m \lim_{h\to 0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+\ldots\right]$$
$$=(cx+d)^m\left[\frac{mc}{cx+d}+0\right]$$
$$=\frac{mc(cx+d)^m}{(cx+d)}$$
$$=mc(cx+d)^{m-1}$$
$$\frac{d}{dx}(cx+d)^m = mc(cx+d)^{m-1} \qquad \ldots(2)$$
$$\text{Similarly, } \frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1} \qquad \ldots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$f'(x)=(ax+b)^n\left\{mc(cx+d)^{m-1}\right\}+(cx+d)^m\left\{na(ax+b)^{n-1}\right\}$$
$$=(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right]$$

Question 14:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): sin (x + a)

Answer:

Let $f(x)=\sin(x+a)$

$$f(x+h)=\sin(x+h+a)$$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right] \\
&= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right] \\
&= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} && \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right] \\
&= \cos\left(\frac{2x+2a}{2}\right)\times 1 && \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right] \\
&= \cos(x+a)
\end{aligned}
$$

Question 15:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): cosec x cot x

Answer:

Let $f(x) = \operatorname{cosec} x \cot x$

By Leibnitz product rule,

$$f'(x) = \operatorname{cosec} x(\cot x)' + \cot x(\operatorname{cosec} x)' \qquad \text{...(1)}$$

Let $f_1(x) = \cot x$. Accordingly, $f_1(x+h) = \cot(x+h)$

By first principle,

$$
\begin{aligned}
f_1'(x) &= \lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}\\
&= \lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]\\
&= \frac{1}{\sin x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]\\
&= \frac{-1}{\sin x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)\\
&= \frac{-1}{\sin x}.1.\left(\frac{1}{\sin(x+0)}\right)\\
&= \frac{-1}{\sin^2 x}\\
&= -\operatorname{cosec}^2 x
\end{aligned}
$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \qquad \ldots(2)$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle,

$$
\begin{aligned}
f_2'(x) &= \lim_{h\to 0}\frac{f_2(x+h)-f_2(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right]
\end{aligned}
$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{-1}{\sin x} \cdot \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cdot \cot x$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$f'(x) = \operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right) + \cot x\left(-\operatorname{cosec} x \cot x\right)$$

$$= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

Question 16:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{\cos x}{1+\sin x}$

Answer:

Let $f(x) = \frac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x)=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$

$$=\frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{-1}{(1+\sin x)}$$

Question 17:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{\sin x+\cos x}{\sin x-\cos x}$

Answer:

Let $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$

By quotient rule,

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$=\frac{-\left[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x\right]}{(\sin x-\cos x)^2}$$

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$=\frac{-2}{(\sin x-\cos x)^2}$$

Question 18:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{\sec x-1}{\sec x+1}$

Answer:

Let $f(x)=\frac{\sec x-1}{\sec x+1}$

$$f(x)=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$$

By quotient rule,

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2}=\frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$=\frac{2\sin x\sec^2 x}{(\sec x+1)^2}$$

$$=\frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$=\frac{2\sec x\tan x}{(\sec x+1)^2}$$

Question 19:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\sin^n x$

Answer:

Let $y = \sin^n x$.

Accordingly, for n = 1, y = sin x.

$$\therefore \frac{dy}{dx}=\cos x,\ \text{i.e.,}\ \frac{d}{dx}\sin x=\cos x$$

For n = 2, $y = \sin^2 x$.

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\sin x\sin x)$$

$$=(\sin x)'\sin x+\sin x(\sin x)' \qquad [\text{By Leibnitz product rule}]$$

$$=\cos x\sin x+\sin x\cos x$$

$$=2\sin x\cos x \qquad ...(1)$$

For n = 3, $y = \sin^3 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\sin x \sin^2 x\right)$$
$$= (\sin x)' \sin^2 x + \sin x\left(\sin^2 x\right)' \qquad \text{[By Leibnitz product rule]}$$
$$= \cos x \sin^2 x + \sin x\left(2\sin x \cos x\right) \qquad \text{[Using (1)]}$$
$$= \cos x \sin^2 x + 2\sin^2 x \cos x$$
$$= 3\sin^2 x \cos x$$

We assert that $\frac{d}{dx}\left(\sin^n x\right) = n\sin^{(n-1)} x \cos x$

Let our assertion be true for n = k.

i.e., $\frac{d}{dx}\left(\sin^k x\right) = k\sin^{(k-1)} x \cos x$ ...(2)

Consider

$$\frac{d}{dx}\left(\sin^{k+1} x\right) = \frac{d}{dx}\left(\sin x \sin^k x\right)$$
$$= (\sin x)' \sin^k x + \sin x\left(\sin^k x\right)' \qquad \text{[By Leibnitz product rule]}$$
$$= \cos x \sin^k x + \sin x\left(k\sin^{(k-1)} x \cos x\right) \qquad \text{[Using (2)]}$$
$$= \cos x \sin^k x + k\sin^k x \cos x$$
$$= (k+1)\sin^k x \cos x$$

Thus, our assertion is true for n = k + 1.

Hence, by mathematical induction, $\frac{d}{dx}\left(\sin^n x\right) = n\sin^{(n-1)} x \cos x$

Question 20:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{a+b\sin x}{c+d\cos x}$

Answer:

Let $f(x) = \frac{a+b\sin x}{c+d\cos x}$

By quotient rule,

$$f'(x)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd\left(\cos^2 x+\sin^2 x\right)}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

Question 21:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{\sin(x+a)}{\cos x}$

Answer:

Let $f(x)=\frac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)(-\sin x)}{\cos^2 x} \quad \text{... (i)}$$

Let $g(x)=\sin(x+a)$. Accordingly, $g(x+h)=\sin(x+h+a)$

By first principle,

$$g'(x)=\lim_{h\to0}\frac{g(x+h)-g(x)}{h}$$
$$=\lim_{h\to0}\frac{1}{h}\left[\sin(x+h+a)-\sin(x+a)\right]$$
$$=\lim_{h\to0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$
$$=\lim_{h\to0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$
$$=\lim_{h\to0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$
$$=\lim_{h\to0}\cos\left(\frac{2x+2a+h}{2}\right).\lim_{\frac{h}{2}\to0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\qquad\left[\text{As } h\to0\Rightarrow\frac{h}{2}\to0\right]$$
$$=\left(\cos\frac{2x+2a}{2}\right)\times1\qquad\left[\lim_{h\to0}\frac{\sin h}{h}=1\right]$$
$$=\cos(x+a)\qquad\ldots\text{(ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{\cos x\cdot\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$
$$=\frac{\cos(x+a-x)}{\cos^2 x}$$
$$=\frac{\cos a}{\cos^2 x}$$

Question 22:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): x^4 (5 sin x – 3 cos x)

Answer:

Let $f(x)=x^4(5\sin x-3\cos x)$

By product rule,

$$f'(x) = x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$
$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$
$$= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3)$$
$$= x^3\left[5x\cos x + 3x\sin x + 20\sin x - 12\cos x\right]$$

Question 23:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $(x^2 + 1)\cos x$

Answer:

Let $f(x) = (x^2 + 1)\cos x$

By product rule,

$$f'(x) = (x^2 + 1)\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2 + 1)$$
$$= (x^2 + 1)(-\sin x) + \cos x(2x)$$
$$= -x^2 \sin x - \sin x + 2x\cos x$$

Question 24:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $(ax^2 + \sin x)(p + q\cos x)$

Answer:

Let $f(x) = (ax^2 + \sin x)(p + q\cos x)$

By product rule,

$$f'(x) = (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x)$$
$$= (ax^2 + \sin x)(-q\sin x) + (p + q\cos x)(2ax + \cos x)$$
$$= -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x)$$

Question 25:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $(x + \cos x)(x - \tan x)$

Answer:

Let $f(x)=(x+\cos x)(x-\tan x)$

By product rule,

$$\begin{aligned}
f'(x)&=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)\\
&=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)\\
&=(x+\cos x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)(1-\sin x) \qquad \text{... (i)}
\end{aligned}$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan(x+h)$

By first principle,

$$\begin{aligned}
g'(x)&=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)\\
&=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]\\
&=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]\\
&=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]\\
&=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]\\
&=\frac{1}{\cos x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)\\
&=\frac{1}{\cos x}.1.\frac{1}{\cos(x+0)}\\
&=\frac{1}{\cos^2 x}\\
&=\sec^2 x \qquad \text{... (ii)}
\end{aligned}$$

Therefore, from (i) and (ii), we obtain

$$\begin{aligned}
f'(x)&=(x+\cos x)(1-\sec^2 x)+(x-\tan x)(1-\sin x)\\
&=(x+\cos x)(-\tan^2 x)+(x-\tan x)(1-\sin x)\\
&=-\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)
\end{aligned}$$

Question 26:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Answer:

Let $f(x)=\frac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35\left(\cos^2 x+\sin^2 x\right)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Question 27:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers):

$$\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$$

Answer:

Let $f(x)=\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x)=\cos\frac{\pi}{4}\cdot\left[\frac{\sin x\frac{d}{dx}\left(x^2\right)-x^2\frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$=\cos\frac{\pi}{4}\cdot\left[\frac{\sin x\cdot 2x-x^2\cos x}{\sin^2 x}\right]$$

$$=\frac{x\cos\frac{\pi}{4}\left[2\sin x-x\cos x\right]}{\sin^2 x}$$

Question 28:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{x}{1+\tan x}$

Answer:

Let $f(x)=\frac{x}{1+\tan x}$

$$f'(x)=\frac{(1+\tan x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x)=\frac{(1+\tan x)-x\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \quad \text{... (i)}$$

Let $g(x)=1+\tan x$. Accordingly, $g(x+h)=1+\tan(x+h)$.

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1+\tan x)=\sec^2 x \qquad \text{... (ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$$

Question 29:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): (x + sec x) (x – tan x)

Answer:

Let $f(x)=(x+\sec x)(x-\tan x)$

By product rule,

$$f'(x) = (x+\sec x)\frac{d}{dx}(x-\tan x) + (x-\tan x)\frac{d}{dx}(x+\sec x)$$

$$= (x+\sec x)\left[\frac{d}{dx}(x) - \frac{d}{dx}\tan x\right] + (x-\tan x)\left[\frac{d}{dx}(x) + \frac{d}{dx}\sec x\right]$$

$$= (x+\sec x)\left[1 - \frac{d}{dx}\tan x\right] + (x-\tan x)\left[1 + \frac{d}{dx}\sec x\right] \qquad \text{... (i)}$$

Let $f_1(x) = \tan x$, $f_2(x) = \sec x$

Accordingly, $f_1(x+h) = \tan(x+h)$ and $f_2(x+h) = \sec(x+h)$

$$f_1'(x) = \lim_{h\to 0}\left(\frac{f_1(x+h) - f_1(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\tan(x+h) - \tan x}{h}\right)$$

$$= \lim_{h\to 0}\left[\frac{\tan(x+h) - \tan x}{h}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$= \left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1\times\frac{1}{\cos^2 x} = \sec^2 x$$

$$\Rightarrow \frac{d}{dx}\tan x = \sec^2 x \qquad \text{... (ii)}$$

$$f_2'(x) = \lim_{h \to 0}\left(\frac{f_2(x+h) - f_2(x)}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\sec(x+h) - \sec x}{h}\right)$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \frac{1}{\cos x}.\lim_{h \to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x}.\lim_{h \to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x}.\lim_{h \to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\cos(x+h)}\right]$$

$$= \sec x.\frac{\left\{\lim_{h \to 0}\sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2} \to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\lim_{h \to 0}\cos(x+h)}$$

$$= \sec x.\frac{\sin x.1}{\cos x}$$

$$\Rightarrow \frac{d}{dx}\sec x = \sec x \tan x \qquad \text{.. (iii)}$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

Question 30:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r ands are fixed non-zero constants and m and n are integers): $\frac{x}{\sin^n x}$

Answer:

Let $f(x)=\frac{x}{\sin^n x}$

By quotient rule,

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx}\sin^n x=n\sin^{n-1}x\cos x$

Therefore,

$$\begin{aligned}f'(x)&=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}\\&=\frac{\sin^n x.1-x\left(n\sin^{n-1}x\cos x\right)}{\sin^{2n} x}\\&=\frac{\sin^{n-1}x\left(\sin x-nx\cos x\right)}{\sin^{2n} x}\\&=\frac{\sin x-nx\cos x}{\sin^{n+1} x}\end{aligned}$$
