EXERCISE 3.3

1. Prove that: 1) $\sin^2\pi/6 + \cos^2\pi/3 - \tan^2\pi/4 = -1/2$

Solution:

We know that :

cos $\pi/3 = 1/2$, sin $\pi/6 = 1/2$, and tan $\pi/4 = 1$ (by trigonometric table).

Substituting these values in the LHS,

LHS = $\sin^2\pi/6 + \cos^2\pi/3 - \tan^2\pi/4$

$= (1/2)^2 + (1/2)^2 - (1)^2$

$= (1/4) + (1/4) - 1$

$= -1/2$

= RHS.

2. Prove that: 2. $2\sin^2\pi/6 + \operatorname{cosec}^2 7\pi/6 \cos^2\pi/3 = 3/2$

Solution:

We know that:

cos $\pi/3 = 1/2$, sin $\pi/6 = 1/2$ and cosec $7\pi/6 = -2$ (by trigonometric table).

Substituting these values in the LHS,

LHS = $2\sin^2\pi/6 + \operatorname{cosec}^2 7\pi/6 \cos^2\pi/3$

$= 2 \times (1/2)^2 + (-2)^2 \times (1/2)^2$

$= 2 \times (1/4) + 4 \times (1/4)$

$= (1/2) + 1$

$= 3/2$

= RHS

3. Prove that: 3) $\cot^2\pi/6 + \operatorname{cosec} 5\pi/6 + 3\tan^2\pi/6 = 6$

Solution:

We know that:

cos $\pi/6 = \sqrt{3}$, cosec $5\pi/6 = 2$ and tan $\pi/6 = 1/\sqrt{3}$ (by trigonometric table).

Substituting these values in the LHS,

LHS = $\cot^2\pi/6 + \text{cosec } 5\pi/6 + 3\tan^2\pi/6 = 6$

$= (\sqrt{3})^2 + 2 + 3 \times (1/\sqrt{3})^2$

$= 3 + 2 + [3 \times (1/3)]$

$= 3 + 2 + 1$

$= 6$

= RHS

4. Prove that: 4) $2\sin^2 3\pi/4 + 2\cos^2\pi/4 + 2\sec^2\pi/3 = 10$

Solution:

We know that:

cos $\pi/4 = 1/\sqrt{2}$, sin $3\pi/4 = 1/\sqrt{2}$ and sec $\pi/3 = 2$ (by trigonometric table).

Substituting these values in the LHS,

LHS = $2\sin^2 3\pi/4 + 2\cos 2\pi/4 + 2\sec^2\pi/3$

$= 2 \times (1/\sqrt{2})^2 + 2 \times (1/\sqrt{2})^2 + 2 \times (2)^2$

$= 2 \times (1/2) + 2 \times (1/2) + 2 \times 4$

$= 1 + 1 + 8$

$= 10$

= RHS

5.Find the value of: (i) sin 75° (ii) tan 15°

Solution:

(i) sin 75°

$\sin 75° = \sin (45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$ [$\because \sin (x+y) = \sin x \cos y + \cos x \sin y$]

$= (1/\sqrt{2}) \times (\sqrt{3}/2) + (1/\sqrt{2}) \times (1/2)$ (by trigonometric table)

$= \sqrt{3}/(2\sqrt{2}) + 1/(2\sqrt{2})$

$= (\sqrt{3}+1)/(2\sqrt{2})$

(ii) $\tan 15°$

$\tan 15° = \tan (45° - 30°)$

$= (\tan 45° - \tan 30°) / (1 + \tan 45° \tan 30°)$

[$\because \tan (x-y) = (\tan x - \tan y) / (1 + \tan x \tan y)$]

$= [1-(1/\sqrt{3})] / [1+ 1 \times (1/\sqrt{3})]$ (by trigonometric table)

$= [(\sqrt{3}-1)/\sqrt{3}] / [\sqrt{3}+1/\sqrt{3}]$

$= (\sqrt{3}-1)/(\sqrt{3}+1)$

$= [(\sqrt{3}-1)/(\sqrt{3}+1)] \times [(\sqrt{3}-1)/(\sqrt{3}-1)]$ [By rationalizing the denominator]

$= (\sqrt{3}-1)^2 / (3-1)$

$= (3+1-2\sqrt{3}) / 2$

$= 2(2-\sqrt{3})/2$

$= 2-\sqrt{3}$

Prove the following:

6.$\cos(\pi/4 - x) \cos(\pi/4 - y) - \sin(\pi/4 - x) \sin(\pi/4 - y) = \sin(x+y)$

Solution:

LHS $= \cos (\pi/4 - x) \cos(\pi/4 - y) - \sin (\pi/4 - x) \sin(\pi/4 - y)$

$= \cos (\pi/4 - x + \pi/4 - y)$ (Using the sum and difference identities, $\cos x \cos y - \sin x \sin y = \cos (x + y)$)

$= \cos (\pi/2 - (x+y))$

$= \sin(x + y)$ [Using cofunction identities, $\cos(\pi/2 - A) = \sin A$]

= RHS

7. Prove the following: $\tan(\pi/4 + x) / \tan(\pi/4 - x) = [(1 + \tan x) / (1 - \tan x)]^2$

Solution:

LHS = $\tan(\pi/4 + x) / \tan(\pi/4 - x)$

= $[(\tan \pi/4 + \tan x) / (1 - \tan \pi/4 \tan x)] / [(\tan \pi/4 - \tan x) / (1 + \tan \pi/4 \tan x)]$

[because $\tan(A \pm B) = (\tan A \pm \tan B) / (1 \mp \tan A \tan B)$]

= $[(1 + \tan x) / (1 - \tan x)] / [(1 - \tan x)/(1 + \tan x)]$ [(by trigonometric table, $\tan \pi/4 = 1$]

= $[(1 + \tan x) / (1 - \tan x)] \times [(1 + \tan x)/(1 - \tan x)]$

= $[(1 + \tan x) / (1 - \tan x)]^2$

= RHS

8. $[\cos(\pi + x) \cos(-x)] / [\sin(\pi - x) \cos(\pi/2 + x)] = \cot^2 x$

Solution:

LHS :

$[\cos(\pi + x) \cos(-x)] / [\sin(\pi - x) \cos(\pi/2 + x)]$

= $[(-\cos x) \times (\cos x)] / [(\sin x) \times (-\sin x)]$

[Since $\cos(\pi + x) = -\cos x$, $\cos(-x) = \cos x$, $\cos(\pi/2 + x) = -\sin x$, and $\sin(\pi-x) = \sin x$]

= $-\cos^2 x / -\sin^2 x$

= $\cot^2 x$

[because $\cot x = \cos x / \sin x$]

= RHS

9. Prove the following: $\cos(3\pi/2 + x) \cos(2\pi + x) [\cot(3\pi/2 - x) + \cot(2\pi + x)] = 1$

Solution:

LHS :

$\cos(3\pi/2 + x) \cos(2\pi + x) [\cot(3\pi/2 - x) + \cot(2\pi + x)]$

= $\cos(\pi + \pi/2 + x) \cos(2\pi + x) [\cot(3\pi/2 - x) + \cot(2\pi + x)]$

= -cos (π/2 + x) cos x [tan x + cot x]

[Since cos (2nπ + A) = cos A, cot(2nπ + A) = cot A, cos (π + A) = -cos A, and cot (π/2 - A) = tan A]

= - (-sin x) cos x [tan x + cot x]

[Beacuse cos (π/2 + A) = -sin A]

= sin x cos x [(sin x / cos x) + (cos x / sin x)]

= sin x cos x [($\sin^2 x + \cos^2 x$) / (sin x cos x)]

= $\sin^2 x + \cos^2 x$

= 1 (By trigonometric identity)

= RHS

10. Prove the following: sin (n + 1)x sin (n + 2)x + cos (n + 1)x cos (n + 2)x = cos x

Solution:

LHS = sin (n + 1)x sin (n + 2)x + cos (n + 1)x cos (n + 2)x

= cos (n + 2)x cos (n + 1)x + sin (n + 2)x sin (n + 1)x

= cos [(n + 2)x - (n + 1)x]

[By trigonometric formulas, cos (A - B) = cos A cos B + sin A sin B]

= cos [(n + 2 - n - 1)x]

= cos x

= RHS

11.Prove the following: cos (3π/4 + x) - cos (3π/4 - x) = -√2 sin x

Solution:

LHS = cos (3π/4 + x) - cos (3π/4 - x)

= -2 sin [{(3π/4 + x) + (3π/4 - x)} / 2] sin [{(3π/4 + x) - (3π/4 - x)} / 2]

[Since cos A - cos B = -2sin [(A + B) / 2] sin [(A - B) / 2]]

= -2sin (3π/4) sin (2x/2)

= -2sin (π - π/4) sin x

= -2sin (π/4) sin x

[As sin (π - A) = sin A]

= -2 × 1/√2 × sin x

(by trigonometric table)

= -√2 sin x

= RHS

12. Prove the following: $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

Solution:

LHS = $\sin^2 6x - \sin^2 4x$

= (sin 6x + sin 4x) (sin 6x - sin 4x)

[Using $a^2 - b^2$ formula]

= [2sin {(6x + 4x) / 2} cos {(6x - 4x) / 2}] × [2 sin {(6x - 4x) / 2} cos {(6x +4x) / 2}]

[Since sin A ± sin B = 2sin [(A ± B) / 2] cos [(A ∓ B) / 2]]

= [2sin 5x cos x] × [2 sin x cos 5x]

= [2sin 5x cos 5x] × [2 sin x cos x]

= [sin (5x + 5x) + sin (5x - 5x)] × [sin (x + x) + sin (x - x)]

[Since 2sin A cos B = sin (A + B) + sin (A - B)]

= [sin 10x + sin 0] × [sin 2x + sin 0]

= sin 10x × sin 2x

[by trigonometric table sin 0 = 0]

= sin 2x sin 10x

= RHS

13. Prove the following: $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

Solution:

LHS = $\cos^2 2x - \cos^2 6x$

= $(\cos 2x + \cos 6x)(\cos 2x - \cos 6x)$

[Using $a^2 - b^2$ formula]

= $[2\cos \{(2x + 6x) / 2\} \cos \{(2x - 6x) / 2\}] \times [-2 \sin \{(2x + 6x) / 2\} \sin \{(2x - 6x) / 2\}]$

{Since $\cos A + \cos B = 2\cos [(A + B) / 2] \cos [(A - B) / 2]$ and $\cos A - \cos B = -2\sin [(A + B) / 2] \sin [(A - B) / 2]$}

= $[2\cos 4x \cos (-2x)] \times [-2\sin 4x \sin (-2x)]$

= $[2\cos 4x \cos 2x] \times [-2\sin 4x (-\sin 2x)]$

[By trigonometric formulas, $\cos (-A) = \cos A$ and $\sin (-A) = -\sin A$]

= $[2\cos 4x \cos 2x] \times [2\sin 4x \sin 2x]$

= $[2\cos 4x \sin 4x] \times [2\cos 2x \sin 2x]$

= $[\sin (4x + 4x) - \sin (4x - 4x)] \times [\sin (2x + 2x) - \sin (2x - 2x)]$

[Since $2\cos A \sin B = \sin (A + B) - \sin (A - B)$]

= $[\sin 8x + \sin 0] \times [\sin 4x + \sin 0]$

= $\sin 8x \times \sin 4x$

[by trigonometric table $\sin 0 = 0$]

= $\sin 4x \sin 8x$

= RHS

14. Prove the following: $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

Solution:

LHS = $\sin 2x + 2\sin 4x + \sin 6x$

= $[\sin 2x + \sin 6x] + 2\sin 4x$

= $[2\sin (2x + 4x)/2 \cos (2x - 6x)/2] + 2\sin 4x$

[Since $\sin A + \sin B = 2\sin [(A + B) / 2] \cos [(A - B) / 2]$

$= [2\sin 4x \cos (-2x)] + 2\sin 4x$

$= 2\sin 4x \cos 2x + 2 \sin 4x$

[because $\cos (-A) = \cos A$]

$= 2\sin 4x (\cos 2x + 2)$

$= 2\sin 4x (2\cos^2 x - 1 + 1)$

[Using double angle formulas, $\cos 2x = 2\cos^2 x - 1$]

$= 2\sin 4x (2\cos^2 x)$

$= 4\cos^2 x \sin 4x$

= RHS

15. Prove the following: cot 4x(sin 5x + sin 3x) = cot x(sin 5x - sin 3x)

Solution:

LHS = $\cot 4x(\sin 5x + \sin 3x)$

$= \cot 4x [2\sin (5x + 3x)/2 \cos (5x - 3x)/2]$

[Since $\sin A + \sin B = 2\sin [(A + B) / 2] \cos [(A - B) / 2]$]

$= (\cos 4x / \sin 4x) [2 \sin 4x \cos x]$

$= 2\cos 4x \cos x$

$= 2\cos 4x \cos x \times (\sin x / \sin x)$

[multiplied and divided by sin x]

$= (\cos x / \sin x) \times [2\cos 4x \sin x]$

$= \cot x [\sin (4x + x) - \sin (4x - x)]$

[Using product to sum formulas, $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$]

$= \cot x(\sin 5x - \sin 3x)$

= RHS

16. Prove the following: (cos 9x - cos 5x) / (sin 17x - sin 3x) = -sin 2x / cos 10x

Solution:

LHS = (cos 9x - cos 5x) / (sin 17x - sin 3x)

Using trigonometric formula,

Since cos A - cos B = -2sin [(A + B) / 2] sin [(A - B) / 2] and sin A - sin B = 2cos [(A + B) / 2] sin [(A - B) / 2],

= [-2sin (9x + 5x)/2 sin (9x - 5x)/2] / [2cos (17x + 3x)/2 sin (17x - 3x)/2]

= [-2sin 7x sin 2x] / [2cos 10x sin 7x]

= -sin 2x / cos 10x

= RHS

17. Prove the following: (sin 5x + sin 3x) / (cos 5x + cos 3x) = tan 4x

Solution:

LHS = (sin 5x + sin 3x) / (cos 5x + cos 3x)

Since cos A + cos B = 2cos [(A + B) / 2] cos [(A - B) / 2] and sin A + sin B

= 2sin [(A + B) / 2] cos [(A - B) / 2],

= [2sin (5x + 3x)/2 cos (5x - 3x)/2] / [2cos (5x + 3x)/2 cos (5x - 3x)/2]

= sin 4x / cos 4x

= tan 4x

= RHS

18. Prove the following: (sin x - sin y) / (cos x + cos y) = tan (x - y)/2

Solution:

LHS = (sin x - sin y) / (cos x + cos y)

Since sin A - sin B = 2cos [(A + B) / 2] sin [(A - B) / 2] and cos A + cos B = 2cos [(A + B) / 2] cos [(A - B) / 2],

= [2cos (x + y)/2 sin (x - y)/2] / [2cos (x + y)/2 cos (x - y)/2]

= [sin (x - y) / 2] / [cos (x - y) / 2]

= tan (x - y) / 2

= RHS

19. Prove the following: (sin x + sin 3x) / (cos x + cos 3x) = tan 2x

Solution:

LHS = (sin x + sin 3x) / (cos x + cos 3x)

Since sin A + sin B = 2sin [(A + B) / 2] cos [(A - B) / 2]

and cos A + cos B = 2cos [(A + B) / 2] cos [(A - B) / 2,

= [2sin (x + 3x)/2 cos (x - 3x)/2] / [2cos (x + 3x)/2 cos (x - 3x)/2]

= sin 2x / cos 2x

= tan 2x

= RHS

20. Prove the following: (sin x - sin 3x) / ($\sin^2$x - $\cos^2$x) = 2sin x

Solution:

LHS = (sin x - sin 3x) / ($\sin^2$x - $\cos^2$x)

= [2cos (x + 3x)/2 sin (x - 3x)/2] / -[$\cos^2$x - $\sin^2$x]

[Since sin A - sin B = 2 cos [(A + B) / 2] sin [(A - B) / 2]]

= 2cos 2x sin (-x) / -cos 2x

[By double angle formulas, cos 2x = $\cos^2$x - $\sin^2$x]

= 2cos 2x (-sin x) / (- cos 2x)

[As sin (-A) = - sin A]

= 2sin x theboardstudy.com

= RHS

21. Prove the following: (cos 4x + cos 3x + cos 2x) / (sin 4x + sin 3x + sin 2x) = cot 3x

Solution:

LHS = (cos 4x + cos 3x + cos 2x) / (sin 4x + sin 3x + sin 2x)

= ([cos 4x + cos 2x] + cos 3x) / ([sin 4x + sin 2x] + sin 3x)

Since cos A + cos B = 2cos [(A + B) / 2] cos [(A - B) / 2] and sin A + sin B = 2sin [(A + B) / 2] cos [(A - B) / 2],

= {[2cos (4x + 2x)/2 cos (4x - 2x)/2] + cos 3x} / {[2sin (4x + 2x)/2 cos (4x - 2x)/2] + sin 3x}

= (2cos 3x cos x + cos 3x) / (2sin 3x sin x + sin 3x)

= [cos 3x(2cos x + 1)] / [sin 3x(2cos x + 1)]

= cos 3x / sin 3x

= cot 3x

= RHS

22. Prove the following: cot x cot 2x - cot 2x cot 3x - cot 3x cot x = 1

Solution:

LHS = cot x cot 2x - cot 2x cot 3x - cot 3x cot x

= cot x cot 2x - cot 3x(cot 2x + cot x)

= cot x cot 2x - cot (2x + x) (cot 2x + cot x)

= cot x cot 2x - [(cot 2x cot x - 1) / (cot 2x + cot x)] (cot 2x + cot x)   [As cot (A + B) = (cot A cot B - 1) / (cot A + cot B)]

= cot x cot 2x - [cot 2x cot x - 1]

= cot x cot 2x - cot 2x cot x + 1

= 1

= RHS

23. Prove the following: tan 4x = [4tan x(1 - $\tan^2$x)] / [1 - 6$\tan^2$x + $\tan^4$x]

Solution:

LHS = tan 4x

= tan 2(2x)

= (2tan 2x) / (1 - $\tan^2$2x)   [By double angle formulas, tan 2A = (2tan A) / (1 - $\tan^2$A)]

= [2(2tan x) / (1 - $\tan^2$x)] / [1 - {(2tan x) / (1 - $\tan^2$x)^2}]   [Again by the same formula]

= [(4tan x) / (1 - $\tan^2$x)] / [1 - (4$\tan^2$x) / (1 + $\tan^4$x - 2$\tan^2$x)]   [Since $(a - b)^2 = a^2 + b^2 - 2ab$]

$= [(4\tan x) / (1 - \tan^2 x)] / [(1 + \tan^4 x - 2\tan^2 x - 4\tan^2 x) / (1 + \tan^4 x - 2\tan^2 x)]$

$= [(4\tan x) / (1 - \tan^2 x)] \times [(1 + \tan^4 x - 2\tan^2 x) / (1 + \tan^4 x - 6\tan^2 x)]$

$= [4\tan x(1 - \tan^2 x)^2] / [(1 - \tan^2 x)(1 + \tan^4 x - 6\tan^2 x)]$ [As $a^2 + b^2 - 2ab = (a - b)^2$]

$= [4\tan x(1 - \tan^2 x)] / [1 - 6\tan^2 x + \tan^4 x]$

= RHS

24. Prove the following: $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

Solution:

LHS = $\cos 4x$

$= \cos 2(2x)$

$= 1 - 2\sin^2 2x$ [By double angle formulas, $\cos 2A = 1 - 2\sin^2 A$]

$= 1 - 2(2\sin x \cos x)^2$ [By double angle formulas, $\sin 2A = 2\sin A \cos A$]

$= 1 - 2(4\sin^2 x\cos^2 x)$

$= 1 - 8\sin^2 x \cos^2 x$

= RHS

25. Prove the following: $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

Solution:

LHS = $\cos 6x$

$= \cos 3(2x)$

$= 4\cos^3 2x - 3\cos 2x$ [As $\cos 3A = 4\cos^3 A - 3\cos A$]

$= 4(2\cos^2 x - 1)^3 - 3(2\cos^2 x - 1)$ [By double angle formula, $\cos 2A = 2\cos^2 A - 1$]

$= 4(8\cos^6 x - 1 - 12\cos^4 x + 6\cos^2 x) - 6\cos^2 x + 3$ [Because $(a - b)^3 = a^3 - b^3 - 3a^2 b + 3ab^2$]

$= 32\cos^6 x - 4 - 48\cos^4 x + 24\cos^2 x - 6\cos^2 x + 3$

$= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

= RHS