Class 11^{th}

Maths

Chapter 7

BINOMIALTHEOREM

Exersice7.1

1.Expand each of the expressions in Exercises 1 to 5: $(1 - 2x)^5$

Solution:

By using binomial theorem, the expression $(1 - 2x)^5$ can be expanded as

$(1 - 2x)^5 = {}^5C_0(1)^5 - {}^5C_1 (1)^4 (2x) + {}^5C_2(1)^3 (2x)^2 - {}^5C_3(1)^2 (2x)^3 + {}^5C_4(1) (2x)^4 - {}^5C_5 (2x)^5$

Here, we can calculate the binomial coefficients ${}^5C_0$, ${}^5C_1$, ... using the nCr formula. Then we get

$= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - (32x^5)$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

2. $(2/x - x/2)^5$

Solution:

By using binomial theorem, the expression $(2/x - x/2)^5$ can be expanded as:

$(2/x - x/2)^5 = 5C_0 (2/x)^5 - 5C_1 (2/x)^4 (x/2) + 5C_2(2/x)^3 (x/2)^2 - 5C_3(2/x)^2 (x/2)^3 + 5C_4(2/4) (x/2)^4 - 5C_5 (x/2)^5$

Here, we can calculate the binomial coefficients ${}^5C_0$, ${}^5C_1$, ... using the nCr formula. Then we get

$= 1(32/x^5) - 5(16/x^4)(x/2) + 10(8/x^3) - 10(x^2/4)(x^3/8) + 5(2/x)(x^2/16) - (x^5/32)$

$= 32/x^5 - 40/x^3 + 20/x - 5x + (5/8)x^3 - x^5/32$

3. **(2x - 3)^6**

Solution:

By using binomial theorem, the expression $(2x - 3)^6$ can be expanded as

$(2x - 3)^6 = {}^6C_0\,(2x)^6 - {}^6C_1\,(2x)^5\,(3) + {}^6C_2\,(2x)^4\,(3)^2 - {}^6C_3\,(2x)^3\,(3)^3 + {}^6C_4\,(2x)^2\,(3)^4 - {}^6C_5\,(2x)(3)^5 + {}^6C_6(3)^6$

Here, we can calculate the binomial coefficients ${}^6C_0$, ${}^6C_1$, ... using the nCr formula. Then we get

$= 1(64x^6) - 6(32x^5)(3) + 15(16x^4)\,(9) - 20\,(8x^3)(27) + 15(4x^2)\,(81) - 6\,(2x)\,(243) + 729$

$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

4. **(x/3 + 1/x)^5**

Solution: By using binomial theorem, the expression $(x/3 + 1/x)$ can be expanded as

$(x/3 + 1/x)^5 = 5C_0\,(x/3)^5 - 5C_1\,(x/3)^4\,(1/x) + 5C_2(x/3)^3\,(1/x)^2 - 5C_3(x/3)^2\,(1/x)^3 + 5C_4(x/3)\,(1/x)^4 - 5C_5\,(1/x)^5$

Here, we can calculate the binomial coefficients ${}^5C_0$, ${}^5C_1$, ... using the nCr formula. Then we get

$= 1(x^5/243) + 5\,(x^4/81)\,(1/x) + 10\,(x^3/2)\ x\ 7\,(1/x^2) + 10\,(x^2/9)\,(1/x^3) + 5\,(x/3)\,(1/x^4) + 1/x^5$

$= x^5/243 + 5x^3/81 + 10x/27 + 10/(9x) + +5/(3x^3) + 1/x^5$

5. **(x + 1/x)^6**

Solution: By using binomial theorem, the expression $(x + 1/x)^6$ can be expanded as

$(x + 1/x)^6 = {}^6C_0\,(x)^6 + {}^6C_1\,(x)^5\,(3) + {}^6C_2\,(x)^4\,(1/x)^2 + {}^6C_3\,(x)^3\,(1/x)^3 + {}^6C_4\,(x)^2\,(1/x)^4 - {}^6C_5\,(x)(1/x)^5 + {}^6C_6(1/x)^6$

Here, we can calculate the binomial coefficients ${}^6C_0$, ${}^6C_1$, ... using the nCr formula. Then we get

$= 1(x^6) + 6x^5\,(1/x) + 15x^4\,(1/x^2) + 20x^3\,(1/x^3)$

$= 15x^2\,(1/x^4) + 6(x)\,(1/x^5) + (1/x^6)$

$= x^6 + 6x^4 + 15x^2 + 20 + 15/x^2 + 6x^4 + 1/x^6$

Using binomial theorem, evaluate each of the following:

6. $(102)^5$

Solution:

102 can be expressed as the sum of two numbers whose powers are easier to calculate. Hence, 102 = 100 + 2

Therefore;

$(102)^5 = (100 + 2)^5$

Now we will expand this using binomial theorem.

$= {}^5C_0 (100)^5 + {}^5C_1 (100)^4 (2) + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)(2)^4 + {}^5C_5 (2)^5$

Here, we can calculate the binomial coefficients ${}^5C_0$, ${}^5C_1$, ... using the nCr formula.

= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32

= 11040808032

7. $(96)^3$

Solution:

96 can be expressed as the difference of two numbers whose powers are easier to calculate.

Hence, 96 = 100 - 4

Therefore,

$(96)^3 = (100 - 4)^3$

Now we will expand this using binomial theorem.

$= {}^3C_0 (100)^3 - {}^3C_1 (100)^2 (4) + {}^3C_2 (100) (4)^2 - {}^3C_3 (4)^3$

Here, we can calculate the binomial coefficients ${}^3C_0$, ${}^3C_1$, ... using the nCr formula.

$= (100)^3 - 3 (10000) (4) + 3 (100) (16) - (64)$

= 1000000 - 120000 + 4800 - 64

= 1004800 - 120064

= 884736

8. **$(101)^4$**

Solution:

101 can be expressed as the sum of two numbers whose powers are easier to calculate.

Hence 101 = 100 + 1

Therefore,

$(101)^4 = (100 + 1)^4$

Now we will expand this using binomial theorem.

$= {}^4C_0 (100)^4 + {}^4C_1 (100)^3 (1) + {}^4C_2 (100)^2 (1)^2 + {}^4C_3 (100)(1)^3 + {}^4C_4 (1)^4$

Here, we can calculate the binomial coefficients ${}^4C_0$, ${}^4C_1$, ... using the nCr formula.

$= (100)^4 + 4(100)^3 + 6 (100)^2 + 4 (100) + (1)^4$

= 100000000 + 4000000 + 60000 + 400 + 1

= 104060401

9. **$(99)^5$**

Solution: 99 can be expressed as the difference of two numbers whose powers are easier to calculate.

Hence, 99 = 100 - 1

Therefore,

$(99)^5 = (100 - 1)^5$

Now we will expand this using binomial theorem.

$= {}^5C_0 (100)^5 - {}^5C_1 (100)^4 (1) + {}^5C_2 (100)^3 (1)^2 - {}^5C_3 (100)^2 (1)^3 + {}^5C_4 (100) (1)^4 + {}^5C_5 (1)^5$

Here, we can calculate the binomial coefficients ${}^5C_0$, ${}^5C_1$, ... using the nCr formula.

$= (100)^5 - 5(100)^4 + 10 (100)^3 - 10 (100)^2 + 5(100) - 1$

= 10000000000 - 500000000 + 10000000 -100000 + 500 - 1

= 10010000500 - 500100001

= 9509900499

10.Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000

Solution:

By splitting 1.1 and then applying binomial theorem, the first few terms of $(1.1)^{10000}$ be obtained as

$(1.1)^{10000} = (1 + 0.1)^{10000}$

$= {}^{10000}C_0 + {}^{10000}C_1 (0.1)$ + other positive terms

$= 1 + 10000 \times (0.1)$ + other positive terms

$= 1 + 1000$ + other positive terms

$= 1001$ + other positive terms

> 1000

Hence, $(1.1)^{10000} > 1000$

11. **Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$**

Solution:

Using binomial theorem, we will evaluate the expressions $(a + b)^4$ and $(a - b)^4$.

- $(a + b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^3b + {}^4C_2\, a^2b^2 + {}^4C_3\, ab^3 + {}^4C_4\, b^4$
- $(a - b)^4 = {}^4C_0\, a^4 - {}^4C_1\, a^3b + {}^4C_2\, a^2b^2 - {}^4C_3\, ab^3 + {}^4C_4\, b^4$

Therefore,

$(a + b)^4 - (a - b)^4 = [({}^4C_0\, a^4 + {}^4C_1\, a^3b + {}^4C_2\, a^2b^2 + {}^4C_3\, ab^3 + {}^4C_4\, b^4) - ({}^4C_0\, a^4 - {}^4C_1\, a^3b + {}^4C_2\, a^2b^2 - {}^4C_3\, ab^3 + {}^4C_4\, b^4)]$

$= 2({}^4C_1\, a^3b + {}^4C_3\, ab^3)$

$= 2(4a^3b + 4ab^3)$

$= 8ab\,(a^2 + b^2)$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$

$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8(\sqrt{3})(\sqrt{2})\,((\sqrt{3})^2 + (\sqrt{2})^2)$

$= 8\sqrt{6}\,[3 + 2]$

$= 40\sqrt{6}$

12. Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$

Solution:

Using binomial theorem, the expressions $(x + 1)^6$ and $(x - 1)^6$ can be expanded as

- $(x + 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6$
- $(x - 1)^6 = {}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6$

Therefore,

$(x + 1)^6 + (x - 1)^6 = [({}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6) + ({}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6)]$

$= 2\, [{}^6C_0\, x^6 + {}^6C_2\, x^4 + {}^6C_4\, x^2 + {}^6C_6]$

$= 2\, [x^6 + 15x^4 + 15x^2 + 1]$

Putting $x = \sqrt{2}$

$(x + 1)^6 + (x - 1)^6 = 2\, [(\sqrt{2})^6 + 15\, (\sqrt{2})^4 + 15\, (\sqrt{2})^2 + 1]$

$= 2\, [8 + 15 \times 4 + 15 \times 2 + 1]$

$= 2\, [8 + 60 + 30 + 1]$

$= 2 \times 99$

$= 198$

13. Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer

Solution:

To show that $9^{n+1} - 8n - 9$ is divisible by 64, we need to prove that $9^{n+1} - 8n - 9 = 64k$ where k is some natural number.

By binomial theorem,

$(1 + a)^m = {}^mC_0 + {}^mC_1\, a + {}^mC_2\, a^2 + ... + {}^mC_m\, a^m$

For a = 8 and m = n + 1, we obtain

$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\, a + {}^{n+1}C_2\, a^2 + ... + {}^{n+1}C_{n+1}\, a^{n+1}$

Hence,

$9^{n+1} = 1 + (n + 1)\,(8) + {}^{n+1}C_2\, (8)^2 + ... + {}^{n+1}C_{n+1}\, (8)^{n+1}$

$9^{n+1} = 9 + 8n + (8)^2 [{}^{n+1}C_2 + ... + {}^{n+1}C_{n+1} (8)^{n-1}]$

Therefore, $9^{n+1} - 8n - 9 = 64k$, where $k = [{}^{n+1}C_2 + ... + {}^{n+1}C_{n+1} (8)^{n-1}]$ is a natural number.

Thus, $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer

14. **Prove that $\Sigma_{r=0}^{n} 3^r \, {}^nC_r = 4^n$**

Solution:

We will start with the LHS of the given equation that needs to be proved.

LHS = $\Sigma_{r=0}^{n} 3^r \, {}^nC_r$

$= 3^0 \cdot {}^nC_0 + 3^1 \cdot {}^nC_1 + 3^2 \cdot {}^nC_2 + ... + 3^n \cdot {}^nC_n$

$= {}^nC_0 \, 1^n \, 3^0 + {}^nC_1 \, 1^{n-1} \, 3^1 + {}^nC_2 \, 1^{n-2} \, 3^2 + ... + {}^nC_n \, 1^{n-n} \, 3^n$

Using binomial theorem the above expansion can be written as,

$= (1 + 3)^n$

$= 4^n$

= RHS

Hence proved