

अध्याय-7

समाकलन

(Integrals)

(Important Formulae and Definitions)

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$
2. $\int \frac{1}{x} dx = \log_e |x| + c$
3. $\int e^x dx = e^x + c$
4. $\int a^x dx = \frac{a^x}{\log_e a} + c$
5. $\int \sin x dx = -\cos x + c$
6. $\int \cos x dx = \sin x + c$
7. $\int \sec^2 x dx = \tan x + c$
8. $\int \operatorname{cosec}^2 x dx = -\cot x + c$
9. $\int \sec x \tan x dx = \sec x + c$
10. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$
11. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
12. $\int \frac{1}{\sqrt{1+x^2}} dx = \tan^{-1} x + c$
13. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$
14. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
15. $\int \frac{dx}{ax+b} = \frac{1}{a} \log |ax+b| + c$
16. $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
17. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$
19. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$

20. $\int \tan x \, dx = -\log |\cos x| + c$ या $\log |\sec x| + c$
21. $\int \cot x \, dx = \log |\sin x| + c$
22. $\int \operatorname{cosec} x \, dx = \log \left| \tan \frac{x}{2} \right| + c = \log |\operatorname{cosec} x - \cot x| + c$
23. $\int \sec x \, dx = \log |\sec x + \tan x| + c = \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + c$
24. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$
25. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + c$
26. $\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2} \left[x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right] + c$
27. $\int \sqrt{a^2 + x^2} \, dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log_e |x + \sqrt{a^2 + x^2}| + c$
28. $\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log_e |x + \sqrt{x^2 - a^2}| + c$
29. $\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \log_e \left| \frac{x-a}{x+a} \right| + c$, जब $x > a$
30. $\int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \log_e \left| \frac{x+a}{x-a} \right| + c$, जब $x < a$
31. $\int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}}$, जब $4ac > b^2$
 $= -\frac{1}{\sqrt{b^2 - 4ac}} \log_e \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}$, जब $4ac < b^2$.

प्रश्नावली 7.1

निम्नलिखित फलनों के प्रतिअवकलज (समाकलन) निरीक्षण विधि द्वारा ज्ञात कीजिए :

प्रश्न 1. $\sin 2x$.

हल : हम जानते हैं कि

$$\frac{d}{dx} \cos 2x = -2 \sin 2x$$

$$\therefore -\frac{1}{2} \frac{d}{dx} \cos 2x = \sin 2x$$

$$\therefore \int \sin 2x \, dx = -\frac{1}{2} \cos 2x + C.$$

उत्तर

प्रश्न 2. $\cos 3x$.

हल : हम जानते हैं कि

$$\frac{d}{dx} \sin 3x = 3 \cos 3x$$

$$\therefore \cos 3x = \frac{d}{dx} \left(\frac{1}{3} \sin 3x \right)$$

$$\therefore \int \cos 3x \, dx = \frac{1}{3} \sin 3x + C. \quad \text{उत्तर}$$

प्रश्न 3. e^{2x} .

हल : हम जानते हैं कि

$$\frac{d}{dx} e^{2x} = 2e^{2x}$$

$$e^{2x} = \frac{d}{dx} \left(\frac{1}{2} e^{2x} \right)$$

$$\therefore \int e^{2x} \, dx = \frac{1}{2} e^{2x} + C. \quad \text{उत्तर}$$

प्रश्न 4. $(ax + b)^2$.

हल : हम जानते हैं कि

$$\frac{d}{dx} (ax + b)^3 = 3a(ax + b)^2$$

$$\text{या} \quad (ax + b)^2 = \frac{d}{dx} \left(\frac{1}{3a} (ax + b)^3 \right)$$

$$\text{या} \quad \int (ax + b)^2 \, dx = \frac{1}{3a} (ax + b)^3 + C. \quad \text{उत्तर}$$

प्रश्न 5. $\sin 2x - 4e^{3x}$

हल : हम जानते हैं कि

$$\frac{d}{dx} \cos 2x = -2 \sin 2x$$

$$\text{या} \quad \sin 2x = \frac{d}{dx} \left(-\frac{1}{2} \cos 2x \right)$$

$$\therefore \int \sin 2x \, dx = -\frac{1}{2} \cos 2x + C_1 \quad \dots(i)$$

$$\text{और} \quad \frac{d}{dx} (e^{3x}) = 3e^{3x} \text{ या } \frac{d}{dx} \left(\frac{1}{3} e^{3x} \right) = e^{3x}$$

$$\therefore \int e^{3x} \, dx = \frac{1}{3} e^{3x} + C_2 \quad \dots(ii)$$

समी. (i) तथा (ii) से,

$$\int (\sin 2x - 4e^{3x}) \, dx = -\frac{1}{2} \cos 2x - \frac{4}{3} e^{3x} + C \quad [\because C = C_1 + C_2] \quad \text{उत्तर}$$

निम्नलिखित समाकलनों को ज्ञात कीजिए—

प्रश्न 6. $\int (4e^{3x} + 1) dx$.

हल :
$$\begin{aligned}\int (4e^{3x} + 1) dx &= 4 \int e^{3x} dx + \int dx \\ &= \frac{4}{3} e^{3x} + x + C.\end{aligned}$$

उत्तर

प्रश्न 7. $\int x^2 \left(1 - \frac{1}{x^2}\right) dx$.

हल :
$$\begin{aligned}\int x^2 \left(1 - \frac{1}{x^2}\right) dx &= \int \left(x^2 - x^2 \cdot \frac{1}{x^2}\right) dx = \int (x^2 - 1) dx \\ &= \int x^2 dx - \int dx \\ &= \frac{x^3}{3} - x + C.\end{aligned}$$

उत्तर

प्रश्न 8. $\int (ax^2 + bx + c) dx$.

हल :
$$\begin{aligned}\int (ax^2 + bx + c) dx &= a \int x^2 dx + b \int x dx + c \int dx \\ &= a \cdot \frac{x^3}{3} + b \cdot \frac{x^2}{2} + cx + C \\ &= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C.\end{aligned}$$

उत्तर

प्रश्न 9. $\int (2x^2 + e^x) dx$.

हल :
$$\begin{aligned}2 \int x^2 dx + \int e^x dx &= 2 \times \frac{x^3}{3} + e^x + c \\ &= \frac{2}{3} x^3 + e^x + c.\end{aligned}$$

उत्तर

प्रश्न 10. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$.

हल :
$$\begin{aligned}\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx &= \int \left[(\sqrt{x})^2 - 2\sqrt{x} \cdot \frac{1}{\sqrt{x}} + \left(\frac{1}{\sqrt{x}}\right)^2\right] dx \\ &= \int \left(x - 2 + \frac{1}{x}\right) dx = \int x dx - 2 \int dx + \int \frac{1}{x} dx \\ &= \frac{x^2}{2} - 2x + \log |x| + C.\end{aligned}$$

उत्तर

प्रश्न 11. $\int \frac{x^3 + 5x^2 - 4}{x^2} dx$.

हल :
$$\begin{aligned}\int \frac{x^3 + 5x^2 - 4}{x^2} dx &= \int \left(\frac{x^3}{x^2} + \frac{5x^2}{x^2} - \frac{4}{x^2}\right) dx = \int \left(x + 5 - \frac{4}{x^2}\right) dx \\ &= \int x dx + 5 \int dx - 4 \int \frac{1}{x^2} dx\end{aligned}$$

$$\begin{aligned}
&= \int x \, dx + 5 \int dx - 4 \int x^{-2} \, dx \\
&= \frac{x^2}{2} + 5x - \frac{4x^{-2+1}}{-2+1} + C \\
&= \frac{x^2}{2} + 5x - \frac{4x^{-1}}{-1} + C \\
&= \frac{x^2}{2} + 5x + \frac{4}{x} + C.
\end{aligned}$$

उत्तर

प्रश्न 12. $\int \frac{x^3 + 3x + 4}{\sqrt{x}} \, dx.$

हल :

$$\begin{aligned}
\int \frac{x^3 + 3x + 4}{\sqrt{x}} \, dx &= \int \left(\frac{x^3}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{4}{\sqrt{x}} \right) dx \\
&= \int (x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}) \, dx \\
&= \int x^{\frac{5}{2}} \, dx + 3 \int x^{\frac{1}{2}} \, dx + 4 \int x^{-\frac{1}{2}} \, dx \\
&= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{4x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \\
&= \frac{2}{7} x^{\frac{7}{2}} + 3 \times \frac{2}{3} x^{\frac{3}{2}} + 4 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C \\
&= \frac{2}{7} x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C.
\end{aligned}$$

उत्तर

प्रश्न 13. $\int \frac{x^3 - x^2 + x - 1}{x - 1} \, dx.$

हल :

$$\begin{aligned}
\int \frac{x^3 - x^2 + x - 1}{x - 1} \, dx &= \int \frac{x^2(x - 1) + 1(x - 1)}{x - 1} \, dx \\
&= \int \frac{(x - 1)(x^2 + 1)}{x - 1} \, dx = \int (x^2 + 1) \, dx \\
&= \int x^2 \, dx + \int dx \\
&= \frac{x^3}{3} + x + C.
\end{aligned}$$

उत्तर

प्रश्न 14. $\int (1 - x)\sqrt{x} \, dx.$

हल :

$$\int (1 - x)\sqrt{x} \, dx = \int \left(x^{\frac{1}{2}} - x^{\frac{3}{2}} \right) dx = \int x^{\frac{1}{2}} \, dx - \int x^{\frac{3}{2}} \, dx$$

$$\begin{aligned}
&= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C \\
&= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C.
\end{aligned}$$

उत्तर

प्रश्न 15. $\int \sqrt{x}(3x^2 + 2x + 3) dx$.

हल :

$$\begin{aligned}
\int \sqrt{x}(3x^2 + 2x + 3) dx &= \int (3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}}) dx \\
&= 3 \int x^{\frac{5}{2}} dx + 2 \int x^{\frac{3}{2}} dx + 3 \int x^{\frac{1}{2}} dx \\
&= 3 \cdot \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 2 \cdot \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 3 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \\
&= 3 \times \frac{2}{7} x^{\frac{7}{2}} + 2 \times \frac{2}{5} x^{\frac{5}{2}} + 3 \times \frac{2}{3} x^{\frac{3}{2}} + C \\
&= \frac{6}{7} x^{\frac{7}{2}} + \frac{4}{5} x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C.
\end{aligned}$$

उत्तर

प्रश्न 16. $\int (2x - 3 \cos x + e^x) dx$.

हल :

$$\begin{aligned}
\int (2x - 3 \cos x + e^x) dx &= 2 \int x dx - 3 \int \cos x dx + \int e^x dx \\
&= 2 \cdot \frac{x^2}{2} - 3 \sin x + e^x + C \\
&= x^2 - 3 \sin x + e^x + C.
\end{aligned}$$

उत्तर

प्रश्न 17. $\int (2x^2 - 3 \sin x + 5\sqrt{x}) dx$.

हल :

$$\begin{aligned}
\int (2x^2 - 3 \sin x + 5\sqrt{x}) dx &= 2 \int x^2 dx - 3 \int \sin x dx + 5 \int \sqrt{x} dx \\
&= 2 \times \frac{x^3}{3} - 3(-\cos x) + \frac{5x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \\
&= \frac{2}{3}x^3 + 3 \cos x + 5 \times \frac{2}{3}x^{\frac{3}{2}} + C \\
&= \frac{2}{3}x^3 + 3 \cos x + \frac{10}{3}x^{\frac{3}{2}} + C.
\end{aligned}$$

उत्तर

प्रश्न 18. $\int \sec x (\sec x + \tan x) dx$.

$$\begin{aligned}\text{हल : } \int \sec x (\sec x + \tan x) dx &= \int (\sec^2 x + \sec x \tan x) dx \\ &= \int \sec^2 x dx + \int \sec x \tan x dx \\ &= \tan x + \sec x + C.\end{aligned}$$

उत्तर

प्रश्न 19. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$.

$$\begin{aligned}\text{हल : } \int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx &= \int \frac{\sin^2 x}{\cos^2 x} = \int \tan^2 x dx \\ &= \int (\sec^2 x - 1) dx \\ &= \int \sec^2 x dx - \int dx \\ &= \tan x - x + C.\end{aligned}$$

उत्तर

प्रश्न 20. $\int \frac{2 - 3 \sin x}{\cos^2 x} dx$.

$$\begin{aligned}\text{हल : } \int \frac{2 - 3 \sin x}{\cos^2 x} dx &= \int \frac{2}{\cos^2 x} dx - \int \frac{3 \sin x}{\cos^2 x} dx \\ &= 2 \int \frac{1}{\cos^2 x} dx - 3 \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx \\ &= 2 \int \sec^2 x dx - 3 \int \sec x \tan x dx \\ &= 2 \tan x - 3 \sec x + C.\end{aligned}$$

उत्तर

प्रश्न 21 व 22 में सही उत्तर का चयन कीजिए —

प्रश्न 21. $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ का प्रतिअवकलज है :

- (A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$
(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

$$\begin{aligned}\text{हल : } \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx &= \int x^{1/2} dx + \int x^{-1/2} dx \\ &= \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C \\ &= \frac{2}{3}x^{3/2} + 2x^{1/2} + C\end{aligned}$$

अतः विकल्प (C) सही है।

उत्तर

प्रश्न 22. यदि $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ जिसमें $f(2) = 0$ तो $f(x)$ है :

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$

(B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$

(D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

हल : ∴

$$\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$$

या

$$\left(4x^3 - \frac{3}{x^4}\right) = f(x) \text{ का प्रति अवकलज}$$

∴

$$\begin{aligned} f(x) &= \int \left(4x^3 - \frac{3}{x^4}\right) dx \\ &= 4 \int x^3 dx - 3 \int x^{-4} dx \\ &= 4 \times \frac{x^4}{4} - 3 \times \frac{x^{-3}}{-3} + C \end{aligned}$$

∴

$$\begin{aligned} f(x) &= x^4 + x^{-3} + C \\ f(2) &= 0 \end{aligned}$$

(दिया है)

∴

$$f(2) = (2)^4 + (2)^{-3} + C = 16 + \frac{1}{8} + C$$

$$\frac{129}{8} + C = 0 \text{ या } C = -\frac{129}{8}$$

∴

$$f(x) = x^4 + x^{-3} - \frac{129}{8}$$

$$f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$$

अतः विकल्प (A) सही है।

उत्तर

प्रश्नावली 7.2

1 से 37 तक के प्रत्येक फलन का समाकलन ज्ञात कीजिए—

प्रश्न 1. $\frac{2x}{1+x^2}$

हल : $\int \frac{2x}{1+x^2} dx$

∴ $1+x^2 = t$ रखने पर

∴

$$2x dx = dt$$

अतः
$$\int \frac{2x}{1+x^2} dx = \int \frac{dt}{t} = \log |t| + C$$

$$= \log (1+x^2) + C$$

(t का मान रखने पर) उत्तर

प्रश्न 2. $\frac{(\log x)^2}{x}$

हल :
$$\int \frac{(\log x)^2}{x} dx$$

 $\therefore \log x = t$ रखने पर

$$\therefore \frac{1}{x} dx = dt$$

अतः
$$\int \frac{(\log x)^2}{x} dx = \int t^2 dt = \frac{t^3}{3} + C$$

$$= \frac{1}{3} (\log x)^3 + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 3. $\frac{1}{x + x \log x}$

हल :
$$\int \frac{1}{x + x \log x} dx = \int \frac{1}{x(1 + \log x)} dx$$

 $\therefore 1 + \log x = t$ प्रतिस्थापित करने पर

$$\therefore \frac{1}{x} dx = dt$$

अतः
$$\int \frac{1}{x(1 + \log x)} dx = \int \frac{1}{t} dt = \log |t| + C$$

$$= \log |1 + \log x| + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 4. $\sin x \sin (\cos x)$.

हल :
$$\int \sin x \sin (\cos x) dx$$

 $\therefore \cos x = t$ प्रतिस्थापित करने पर

$$\therefore -\sin x dx = dt$$

अतः
$$-\int \sin t dt = \cos t + C$$

$$= \cos (\cos x) + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 5. $\sin (ax + b) \cos (ax + b)$.

हल :
$$\int \sin (ax + b) \cos (ax + b) dx$$

 $\therefore \sin (ax + b) = t$ रखने पर

$$\therefore a \cos (ax + b) dx = dt$$

अतः
$$\int \sin (ax + b) \cos (ax + b) dx = \frac{1}{a} \int t dt = \frac{1}{a} \cdot \frac{t^2}{2} + C$$

$$= \frac{\sin^2 (ax + b)}{2a} + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 6. $\sqrt{ax+b}$.

हल : $\int \sqrt{ax+b} \, dx$

$\therefore ax+b=t$ प्रतिस्थापित करने पर

$\therefore a \, dx = dt$

अतः
$$\begin{aligned} \int \sqrt{ax+b} \, dx &= \frac{1}{a} \int \sqrt{t} \, dt = \frac{1}{a} \int t^{1/2} \, dt \\ &= \frac{1}{a} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + C = \frac{2}{3a} t^{\frac{3}{2}} + C \\ &= \frac{2}{3a} (ax+b)^{\frac{3}{2}} + C. \quad (t \text{ का मान रखने पर) \text{ उत्तर} \end{aligned}$$

प्रश्न 7. $x\sqrt{x+2}$.

हल :
$$\begin{aligned} \int x\sqrt{x+2} \, dx &= \int (x+2-2)\sqrt{x+2} \, dx \\ &= \int (x+2)^{\frac{3}{2}} \, dx - 2 \int \sqrt{x+2} \, dx \\ &= \frac{(x+2)^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 2 \cdot \frac{(x+2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \\ &= \frac{2}{5} (x+2)^{\frac{5}{2}} - \frac{4}{3} (x+2)^{\frac{3}{2}} + C. \end{aligned}$$

उत्तर

प्रश्न 8. $x\sqrt{1+2x^2}$.

हल : $\int x\sqrt{1+2x^2} \, dx$

$\therefore 1+2x^2=t$ रखने पर

$4x \, dx = dt$

$\therefore$
$$\begin{aligned} \int x\sqrt{1+2x^2} \, dx &= \frac{1}{4} \int \sqrt{t} \, dt \\ &= \frac{1}{4} \cdot \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{1}{6} t^{\frac{3}{2}} + C \\ &= \frac{1}{6} (1+2x^2)^{\frac{3}{2}} + C. \quad (t \text{ का मान रखने पर) \text{ उत्तर} \end{aligned}$$

प्रश्न 9. $(4x+2)\sqrt{x^2+x+1}$.

हल : $\int (4x+2)\sqrt{x^2+x+1} dx = 2 \int (2x+1)\sqrt{x^2+x+1} dx$

$\because x^2+x+1 = t$ रखने पर

$\therefore (2x+1) dx = dt$

$$\begin{aligned} \therefore &= 2 \int t^{\frac{1}{2}} dt = 2 \cdot \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \\ &= \frac{4}{3} t^{\frac{3}{2}} + C = \frac{4}{3} (x^2+x+1)^{\frac{3}{2}} + C. \end{aligned}$$

(t का मान रखने पर) उत्तर

प्रश्न 10. $\frac{1}{x-\sqrt{x}}$.

हल : $\int \frac{1}{x-\sqrt{x}} dx = \int \frac{1}{\sqrt{x}(\sqrt{x}-1)} dx$

$\because \sqrt{x}-1 = t$ रखने पर

$\therefore \frac{1}{2\sqrt{x}} dx = dt$

$$= 2 \int \frac{1}{t} dt = 2 \log |t| + C$$

$$= 2 \log |\sqrt{x}-1| + C. \quad (t \text{ का मान रखने पर) उत्तर}$$

प्रश्न 11. $\frac{x}{\sqrt{x+4}}, x > 0$.

हल :
$$\begin{aligned} \int \frac{x}{\sqrt{x+4}} dx &= \int \frac{x+4-4}{\sqrt{x+4}} dx \\ &= \int \frac{x+4}{\sqrt{x+4}} dx - \int \frac{4}{\sqrt{x+4}} dx \\ &= \int \sqrt{x+4} dx - 4 \int (x+4)^{-\frac{1}{2}} dx \\ &= \frac{2}{3} (x+4)^{\frac{3}{2}} - 4 \cdot 2 (x+4)^{\frac{1}{2}} + C \\ &= \frac{2}{3} \sqrt{x+4} (x+4) - 8\sqrt{x+4} + C \\ &= 2\sqrt{x+4} \left(\frac{x+4}{3} - 4 \right) + C \\ &= \frac{2}{3} \sqrt{x+4} (x+4-12) + C \\ &= \frac{2}{3} \sqrt{x+4} (x-8) + C. \end{aligned}$$

उत्तर

प्रश्न 12. $(x^3 - 1)^{\frac{1}{3}} x^5$.

हल : $\int (x^3 - 1)^{\frac{1}{3}} x^5 dx$

$$= \int (x^3 - 1)^{\frac{1}{3}} x^3 x^2 dx$$

$\therefore x^3 - 1 = t$ या $x^3 = 1 + t$ रखने पर

$$\therefore 3x^2 dx = dt$$

$$= \frac{1}{3} \int t^{\frac{1}{3}} (1 + t) dt$$

$$= \frac{1}{3} \int (t^{\frac{1}{3}} + t^{\frac{4}{3}}) dt$$

$$= \frac{1}{3} \left[\frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + \frac{t^{\frac{4}{3}+1}}{\frac{4}{3}+1} \right] + C$$

$$= \frac{1}{3} \left[\frac{3}{4} t^{\frac{4}{3}} + \frac{3}{7} t^{\frac{7}{3}} \right] + C$$

$$= \frac{1}{4} t^{\frac{4}{3}} + \frac{1}{7} t^{\frac{7}{3}} + C$$

$$= \frac{1}{4} (x^3 - 1)^{\frac{4}{3}} + \frac{1}{7} (x^3 - 1)^{\frac{7}{3}} + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 13. $\frac{x^2}{(2 + 3x^3)^3}$.

हल : $\int \frac{x^2}{(2 + 3x^3)^3} dx$

$\therefore 2 + 3x^3 = t$ रखने पर

$$\therefore 9x^2 dx = dt$$

$$= \frac{1}{9} \int \frac{dt}{t^3} = \frac{1}{9} \int t^{-3} dt$$

$$= \frac{1}{9} \left[\frac{t^{-3+1}}{-3+1} \right] + C$$

$$= -\frac{1}{18} \cdot \frac{1}{t^2} + C$$

$$= -\frac{1}{18(2 + 3x^3)^2} + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 14. $\frac{1}{x(\log x)^m}, x > 0, m \neq 1.$

हल : $\int \frac{1}{x(\log x)^m} dx$

$\therefore \log x = t$ रखने पर

$$\therefore \frac{1}{x} dx = dt$$

$$= \int \frac{1}{t^m} dt = \int t^{-m} dt = \frac{t^{-m+1}}{-m+1} + C$$

$$= \frac{(\log x)^{1-m}}{1-m} + C. \quad (t \text{ का मान रखने पर) उत्तर}$$

प्रश्न 15. $\frac{x}{9-4x^2}.$

हल : $\int \frac{x}{9-4x^2} dx$

$\therefore 9-4x^2 = t$ रखने पर

$$\therefore -8x dx = dt$$

$$= -\frac{1}{8} \int \frac{dt}{t} = -\frac{1}{8} \log |t| + C$$

$$= \frac{1}{8} \log |t|^{-1} + C$$

$$= \frac{1}{8} \log \frac{1}{(9-4x^2)} + C. \quad (t \text{ का मान रखने पर) उत्तर}$$

प्रश्न 16. $e^{2x+3}.$

हल : $\int e^{2x+3} dx$

$\therefore 2x+3 = t$ रखने पर

$$\therefore 2dx = dt$$

$$= \frac{1}{2} \int e^t dt = \frac{1}{2} e^t + C$$

$$= \frac{1}{2} e^{2x+3} + C. \quad (t \text{ का मान रखने पर) उत्तर}$$

प्रश्न 17. $\frac{x}{e^{x^2}}.$

हल : $\int \frac{x}{e^{x^2}} dx$

$\therefore x^2 = t$ रखने पर

$$\therefore 2x dx = dt$$

$$\therefore = \frac{1}{2} \int \frac{dt}{e^t} = \frac{1}{2} \int e^{-t} dt = -\frac{1}{2} e^{-t} + C$$

$$= -\frac{e^{-x^2}}{2} + C$$

$$= -\frac{1}{2e^{x^2}} + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 18. $\frac{e^{\tan^{-1} x}}{1+x^2}.$

हल :

$$\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$$

 $\therefore \tan^{-1} x = t$ रखने पर

$$\therefore \frac{1}{1+x^2} dx = dt$$

$$= \int e^t dt = e^t + C = e^{\tan^{-1} x} + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 19. $\frac{e^{2x} - 1}{e^{2x} + 1}.$

हल :

$$\int \frac{e^{2x} - 1}{e^{2x} + 1} dx = \int \frac{e^x(e^x - e^{-x})}{e^x(e^x + e^{-x})} dt$$

$$= \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

 $\therefore e^x + e^{-x} = t$ रखने पर

$$\therefore (e^x - e^{-x}) dx = dt$$

$$= \int \frac{dt}{t} = \log |t| + C$$

$$= \log(e^x + e^{-x}) + C.$$

(t का मान रखने पर) उत्तर

प्रश्न 20. $\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}.$

हल :

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$$

 $\therefore e^{2x} + e^{-2x} = t$ रखने पर

$$\therefore (2e^{2x} - 2e^{-2x}) dx = dt$$

या $2(e^{2x} - e^{-2x}) dx = dt$

$$= \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log |t| + C$$

$$= \frac{1}{2} \log |e^{2x} + e^{-2x}| + C. \quad (t \text{ का मान रखने पर) उत्तर}$$

प्रश्न 21. $\tan^2(2x - 3).$

हल :

$$\int \tan^2(2x - 3) dx$$

$$= \int [\sec^2(2x-3) - 1] dx$$

$$= \int \sec^2(2x-3) dx - \int dx$$

$$\because 2x-3 = t \text{ रखने पर}$$

$$\therefore 2dx = dt$$

$$= \frac{1}{2} \int \sec^2 t dt - \int dx = \frac{1}{2} \tan t - x + C$$

$$= \frac{1}{2} \tan(2x-3) - x + C. \quad (t \text{ का मान रखने पर) उत्तर}$$

प्रश्न 22. $\sec^2(7-4x)$.

हल : $\int \sec^2(7-4x) dx$

$$\because 7-4x = t \text{ रखने पर}$$

$$\therefore -4 dx = dt$$

$$= -\frac{1}{4} \int \sec^2 t dt = -\frac{1}{4} \tan t + C$$

$$= -\frac{1}{4} \tan(7-4x) + C. \quad (t \text{ का मान रखने पर) उत्तर}$$

प्रश्न 23. $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$.

हल : $\frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

$$\because \sin^{-1} x = t \text{ रखने पर}$$

$$\therefore \frac{1}{\sqrt{1-x^2}} dx = dt$$

$$= \int t dt = \frac{t^2}{2} + C$$

$$= \frac{(\sin^{-1} x)^2}{2} + C.$$

उत्तर

प्रश्न 24. $\frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x}$.

हल : $\int \frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x} dx$

$$\because 2 \sin x + 3 \cos x = t \text{ रखने पर}$$

$$\therefore (2 \cos x - 3 \sin x) dx = dt$$

$$= \frac{1}{2} \int \frac{2 \cos x - 3 \sin x}{3 \cos x + 2 \sin x} dx$$

$$= \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log |t| + C$$

$$= \frac{1}{2} \log |2 \sin x + 3 \cos x| + C.$$

उत्तर

प्रश्न 25. $\frac{1}{\cos^2 x(1 - \tan x)^2}$.

हल : $\int \frac{1}{\cos^2 x(1 - \tan x)^2} dx = \int \frac{\sec^2 x}{(1 - \tan x)^2} dx$

$\therefore 1 - \tan x = t$ रखने पर

$\therefore -\sec^2 x dx = dt$

$$= -\int \frac{1}{t^2} dt = -\int t^{-2} dt = \frac{-t^{-2+1}}{-2+1} + C$$

$$= -\frac{t^{-1}}{-1} + C = \frac{1}{t} + C$$

$$= -\frac{1}{1 - \tan x} + C.$$

उत्तर

प्रश्न 26. $\frac{\cos \sqrt{x}}{\sqrt{x}}$.

हल : $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

$\therefore \sqrt{x} = t$ रखने पर

$\therefore \frac{1}{2\sqrt{x}} dx = dt$

$\therefore$

$$= 2 \int \cos t dt = 2 \sin t + C$$

$$= 2 \sin \sqrt{x} + C.$$

उत्तर

प्रश्न 27. $\sqrt{\sin 2x} \cos 2x$.

हल : $\int \sqrt{\sin 2x} \cos 2x dx$

$\therefore \sin 2x = t$ रखने पर

$\therefore 2 \cos 2x dx = dt$

$$= \frac{1}{2} \int \sqrt{t} dt = \frac{1}{2} \times \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + C = \frac{1}{3} (\sin 2x)^{\frac{3}{2}} + C.$$

उत्तर

प्रश्न 28. $\frac{\cos x}{\sqrt{1 + \sin x}}$.

हल : $\int \frac{\cos x}{\sqrt{1 + \sin x}} dx$

$\therefore \sin x = t$ रखने पर

$\therefore \cos x dx = dt$

$$= \int \frac{dt}{\sqrt{1+t}} = \frac{(1+t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= 2(1+t)^{\frac{1}{2}} + C$$

$$= 2\sqrt{1+\sin x} + C.$$

उत्तर

प्रश्न 29. $\cot x \log \sin x$.

हल : $\int \cot x \log \sin x \, dx$

$\therefore \log \sin x = t$ रखने पर,

$\therefore \cot x \, dx = dt$

$$= \int t \, dt = \frac{t^2}{2} + C$$

$$= \frac{1}{2}(\log \sin x)^2 + C.$$

उत्तर

प्रश्न 30. $\frac{\sin x}{1 + \cos x}$.

हल : $\int \frac{\sin x}{1 + \cos x} \, dx$

$\therefore 1 + \cos x = t$ रखने पर

$\therefore -\sin x \, dx = dt$

$$= -\int \frac{1}{t} \, dt = -\log |t| + C$$

$$= -\log |1 + \cos x| + C$$

$$= \log \left| \frac{1}{1 + \cos x} \right| + C.$$

उत्तर

प्रश्न 31. $\frac{\sin x}{(1 + \cos x)^2}$.

हल : $\int \frac{\sin x}{(1 + \cos x)^2} \, dx$

$\therefore 1 + \cos x = t$ रखने पर,

$\therefore -\sin x \, dx = dt$

$$= -\int \frac{dt}{t^2} = -\frac{t^{-2+1}}{-2+1} + C = \frac{1}{t} + C$$

$$= \frac{1}{1 + \cos x} + C.$$

उत्तर

प्रश्न 32. $\frac{1}{1 + \cot x}$.

हल : $\int \frac{1}{1 + \cot x} \, dx$

$$\begin{aligned}
&= \int \frac{1}{1 + \frac{\cos x}{\sin x}} dx \\
&= \int \frac{1}{\frac{\sin x + \cos x}{\sin x}} dx \\
&= \int \frac{\sin x}{\sin x + \cos x} dx \\
&= \frac{1}{2} \int \left(\frac{2 \sin x}{\sin x + \cos x} \right) dx \\
&= \frac{1}{2} \int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x} dx \\
&= \int \frac{1}{2} dx - \frac{1}{2} \int \frac{\cos x - \sin x}{\sin x + \cos x} dx
\end{aligned}$$

$\therefore \sin x + \cos x = t$ रखने पर,

$$\begin{aligned}
\therefore (\cos x - \sin x) dx &= dt \\
&= \frac{1}{2} \int dx - \frac{1}{2} \int \frac{1}{t} dt \\
&= \frac{1}{2} x - \frac{1}{2} \log |t| + C \\
&= \frac{1}{2} x - \frac{1}{2} \log |\sin x + \cos x| + C.
\end{aligned}$$

उत्तर

प्रश्न 33. $\frac{1}{1 - \tan x}$.

हल :

$$\begin{aligned}
\int \frac{1}{1 - \tan x} dx &= \int \frac{1}{1 - \frac{\sin x}{\cos x}} dx = \int \frac{1}{\frac{\cos x - \sin x}{\cos x}} dx \\
&= \int \frac{\cos x}{\cos x - \sin x} dx = \frac{1}{2} \int \frac{2 \cos x}{\cos x - \sin x} dx \\
&= \frac{1}{2} \int \frac{\cos x - \sin x + \cos x + \sin x}{\cos x - \sin x} dx \\
&= \frac{1}{2} \int dx - \frac{1}{2} \int \frac{-\cos x - \sin x}{\cos x - \sin x} dx \\
&= \frac{1}{2} \int dx - \frac{1}{2} \int \frac{-\sin x - \cos x}{\cos x - \sin x} dx
\end{aligned}$$

$\therefore \cos x - \sin x = t$ रखने पर,

$$\begin{aligned}
\therefore (-\sin x - \cos x) dx &= dt \\
&= \frac{1}{2} \int dx - \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} x - \frac{1}{2} \log |t| + C \\
&= \frac{1}{2} x - \frac{1}{2} \log |\cos x - \sin x| + C.
\end{aligned}$$

उत्तर

प्रश्न 34. $\frac{\sqrt{\tan x}}{\sin x \cos x}$.

हल :
$$\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = \int \frac{\sqrt{\tan x}}{\frac{\sin x}{\cos x} \cos^2 x} dx$$

$$= \int \frac{\sqrt{\tan x}}{\tan x} \sec^2 x dx = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

$\therefore \tan x = t$ रखने पर
 $\therefore \sec^2 x dx = dt$

$$= \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} dt = \int \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= 2\sqrt{t} + C = 2\sqrt{\tan x} + C.$$

उत्तर

प्रश्न 35. $\frac{(1 + \log x)^2}{x}$.

हल :
$$\int \frac{(1 + \log x)^2}{x} dx$$

$\therefore 1 + \log x = t$ रखने पर,

$\therefore \frac{1}{x} dx = dt$

$$= \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}(1 + \log x)^3 + C.$$

उत्तर

प्रश्न 36. $\frac{(x+1)(x+\log x)^2}{x}$.

हल :
$$\int \frac{(x+1)(x+\log x)^2}{x} dx = \int \left(1 + \frac{1}{x}\right)(x+\log x)^2 dx$$

$\therefore x + \log x = t$ रखने पर,

$\therefore \left(1 + \frac{1}{x}\right) dx = dt$

$$= \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}(x + \log x)^3 + C.$$

उत्तर

प्रश्न 37. $\frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}$.

हल :
$$\int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8} dx$$

$\therefore \tan^{-1} x^4 = t$ रखने पर,

$$\therefore \frac{1}{1+x^8} \frac{d}{dx} x^4 = dt$$

$$\text{या } \frac{4x^3}{1+x^8} dx = dt$$

$$\begin{aligned} \text{या } \frac{x^3}{1+x^8} dx &= \frac{1}{4} dt \\ &= \frac{1}{4} \int \sin t \, dt \\ &= \frac{1}{4} (-\cos t) + C \\ &= -\frac{1}{4} \cos (\tan^{-1} x^4) + C. \end{aligned}$$

उत्तर

प्रश्न 38 एवं 39 में सही उत्तर का चयन कीजिए—

प्रश्न 38. $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ बराबर है—

(A) $10^x - x^{10} + C$

(B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$

(D) $\log (10^x + x^{10}) + C$

हल : $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$

$\therefore x^{10} + 10^x = t$ रखने पर

$\therefore (10x^9 + 10^x \log_e 10) dx = dt$

$$= \int \frac{dt}{t} = \log |t| + C$$

$$= \log |x^{10} + 10^x| + C$$

अतः विकल्प (D) सही है।

उत्तर

प्रश्न 39. $\int \frac{dx}{\sin^2 x \cos^2 x}$ बराबर है—

(A) $\tan x + \cot x + C$

(B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$

(D) $\tan x - \cot 2x + C$

हल : $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx$$

$$= \tan x - \cot x + C.$$

अतः विकल्प (B) सही है।

उत्तर

प्रश्नावली 7.3

1 से 22 तक के प्रश्नों में प्रत्येक फलन का समाकलन ज्ञात कीजिए—

प्रश्न 1. $\sin^2 (2x + 5)$.

हल : $\int \sin^2 (2x + 5) dx$ $\left(\because \sin^2 A = \frac{1 - \cos 2A}{2} \right)$

$$= \frac{1}{2} \int [1 - \cos 2(2x + 5)] dx$$

$$= \frac{1}{2} \int [1 - \cos (4x + 10)] dx$$

यहाँ $4x + 10 = t$ रखने पर

$$4dx = dt$$

$$= \frac{1}{2} \int 1 dx - \frac{1}{2} \cdot \frac{1}{4} \int \cos t dt$$

$$= \frac{1}{2} x - \frac{1}{8} \sin t + C$$

$$= \frac{1}{2} x - \frac{1}{8} \sin (4x + 10) + C.$$

उत्तर

प्रश्न 2. $\sin 3x \cos 4x$.

हल : $\int \sin 3x \cos 4x dx = \frac{1}{2} \int 2 \cos 4x \sin 3x dx$

$$= \frac{1}{2} \int \sin 7x - \sin x dx$$

$$[\because 2 \cos A \sin B = \sin (A + B) - \sin (A - B)]$$

$$= \frac{1}{2} \int \sin 7x dx - \frac{1}{2} \int \sin x dx$$

$$= \frac{1}{2} \left(-\frac{\cos 7x}{7} \right) + \frac{1}{2} (\cos x) + C$$

$$= -\frac{1}{14} \cos 7x + \frac{1}{2} \cos x + C.$$

उत्तर

प्रश्न 3. $\cos 2x \cos 4x \cos 6x$.

हल : $\int \cos 2x \cos 4x \cos 6x dx = \frac{1}{2} \int (2 \cos 4x \cos 2x) \cos 6x dx$

$$= \frac{1}{2} \int [\cos 6x + \cos 2x] \cos 6x dx$$

$$[\because 2 \cos A \cos B = \cos (A + B) + \cos (A - B)]$$

$$= \frac{1}{4} \int (2 \cos^2 6x + 2 \cos 6x \cos 2x) dx$$

$$= \frac{1}{4} \int [1 + \cos 12x + \cos 8x + \cos 4x] dx$$

$$\begin{aligned}
&= \frac{1}{4} \left[\int 1 dx + \int \cos 12x dx + \int \cos 8x dx + \int \cos 4x dx \right] \\
&= \frac{1}{4} \left[x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} \right] + C. \quad \text{उत्तर}
\end{aligned}$$

प्रश्न 4. $\sin^3 (2x + 1)$.

हल :
$$\begin{aligned}
\int \sin^3 (2x + 1) dx &= \int \sin^2 (2x + 1) \cdot \sin (2x + 1) dx \\
&= \int [1 - \cos^2 (2x + 1)] \cdot \sin (2x + 1) dx
\end{aligned}$$

मान लीजिए $\cos (2x + 1) = t$

$$- \sin (2x + 1) 2 dx = dt$$

या

$$\sin (2x + 1) dx = \frac{dt}{2}$$

$$= \frac{-1}{2} \int (1 - t^2) dt$$

$$= \frac{-1}{2} \left[\int 1 dt - \int t^2 dt \right]$$

$$= \frac{-1}{2} \left[t - \frac{t^3}{3} \right] + C$$

$$= \frac{-t}{2} + \frac{t^3}{6} + C$$

$$= -\frac{\cos (2x + 1)}{2} + \frac{\cos^3 (2x + 1)}{6} + C. \quad \text{उत्तर}$$

प्रश्न 5. $\sin^3 x \cos^3 x$.

हल :
$$\begin{aligned}
\int \sin^3 x \cos^3 x dx &= \int \sin^2 x \cos^3 x \sin x dx \\
&= \int (1 - \cos^2 x) \cos^3 x \sin x dx
\end{aligned}$$

$\therefore \cos x = t$ रखने पर

$$\therefore -\sin x dx = dt$$

$$= - \int (1 - t^2) t^3 dt = - \int (t^3 - t^5) dt$$

$$= -\frac{t^4}{4} + \frac{t^6}{6} + C$$

$$= \frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x + C. \quad \text{उत्तर}$$

प्रश्न 6. $\sin x \sin 2x \sin 3x$.

हल :
$$\begin{aligned}
&\int (\sin x \sin 2x \sin 3x) dx \\
&= \frac{1}{2} \int (2 \sin x \sin 2x) \sin 3x dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int (\cos x - \cos 3x) \sin 3x \, dx \\
&\quad [\because 2 \sin A \sin B = \cos(A - B) - \cos(A + B)] \\
&= \frac{1}{4} \int [2 \sin 3x \cos x - 2 \sin 3x \cos 3x] \, dx \\
&= \frac{1}{4} \int [\sin 4x + \sin 2x - \sin 6x] \, dx \\
&\quad [\because 2 \sin A \cos B = \sin(A + B) + \sin(A - B)] \\
&= -\frac{1}{4} \left[\frac{\cos 4x}{4} + \frac{\cos 2x}{2} - \frac{\cos 6x}{6} \right] + C \\
&= \frac{1}{4} \left(\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2} \right) + C. \quad \text{उत्तर}
\end{aligned}$$

प्रश्न 7. $\sin 4x \sin 8x$.

हल :

$$\begin{aligned}
\int \sin 4x \sin 8x \, dx &= \frac{1}{2} \int (2 \sin 8x \sin 4x) \, dx \\
&\quad [\because 2 \sin A \sin B = \cos(A - B) - \cos(A + B)] \\
&= \frac{1}{2} \int [\cos 4x - \cos 12x] \, dx \\
&= \frac{1}{2} \left[\int \cos 4x \, dx - \int \cos 12x \, dx \right] \\
&= \frac{1}{2} \left[\frac{1}{4} \sin 4x - \frac{1}{12} \sin 12x \right] + C. \quad \text{उत्तर}
\end{aligned}$$

प्रश्न 8. $\frac{1 - \cos x}{1 + \cos x}$.

हल :

$$\begin{aligned}
\int \frac{1 - \cos x}{1 + \cos x} \, dx &= \int \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \, dx \\
&\quad [\because 1 - \cos A = 2 \sin^2 \frac{A}{2}, 1 + \cos A = 2 \cos^2 \frac{A}{2}] \\
&= \int \tan^2 \frac{x}{2} \, dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) \, dx \\
&= \int \sec^2 \frac{x}{2} \, dx - \int 1 \, dx \\
&= \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + C \\
&= 2 \tan \frac{x}{2} - x + C. \quad \text{उत्तर}
\end{aligned}$$

प्रश्न 9. $\frac{\cos x}{1 + \cos x}$.

हल :

$$\begin{aligned}
 \int \frac{\cos x}{1 + \cos x} dx &= \int \frac{1 + \cos x - 1}{1 + \cos x} dx \\
 &= \int \frac{1 + \cos x}{1 + \cos x} dx - \int \frac{1}{1 + \cos x} dx \\
 &= \int \left(1 - \frac{1}{1 + \cos x} \right) dx = \int \left(1 - \frac{1}{2 \cos^2 \frac{x}{2}} \right) dx \\
 &\quad [\because 1 + \cos 2A = 2 \cos^2 A] \\
 &= \int \left(1 - \frac{1}{2} \sec^2 \frac{x}{2} \right) dx = \int 1 \cdot dx - \frac{1}{2} \int \sec^2 \frac{x}{2} dx \\
 &= x - \frac{1}{2} \cdot \frac{\tan \frac{x}{2}}{\frac{1}{2}} + C = x - \tan \frac{x}{2} + C. \quad \text{उत्तर}
 \end{aligned}$$

प्रश्न 10. $\sin^4 x$.

हल :

$$\begin{aligned}
 \int \sin^4 x dx &= \int (\sin^2 x)^2 dx \\
 &= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx \quad \left[\because \sin^2 x = \frac{1 - \cos 2x}{2} \right] \\
 &= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{4} \int \left(1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) dx \\
 &\quad \left[\because \cos^2 A = \frac{1 + \cos 2A}{2} \right] \\
 &= \frac{1}{8} \int (2 - 4 \cos 2x + \cos 4x + 1) dx \\
 &= \frac{1}{8} \int (3 - 4 \cos 2x + \cos 4x) dx \\
 &= \frac{1}{8} \left[\int 3 dx - 4 \int \cos 2x dx + \int \cos 4x dx \right] \\
 &= \frac{1}{8} \left[3x - 4 \cdot \frac{\sin 2x}{2} + \frac{\sin 4x}{4} \right] + C \\
 &= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C. \quad \text{उत्तर}
 \end{aligned}$$

प्रश्न 11. $\cos^4 2x$.

हल :

$$\int \cos^4 2x dx = \int (\cos^2 2x)^2 dx$$

$$\begin{aligned}
&= \int \left(\frac{1 + \cos 4x}{2} \right)^2 dx \quad \left(\because \cos^2 A = \frac{1 + \cos 2A}{2} \right) \\
&= \frac{1}{4} \int (1 + 2 \cos 4x + \cos^2 4x) dx \\
&= \frac{1}{4} \int \left(1 + 2 \cos 4x + \frac{1 + \cos 8x}{2} \right) dx \\
&= \frac{1}{4} \int \frac{(2 + 4 \cos 4x + 1 + \cos 8x)}{2} dx \\
&= \frac{1}{8} \int (3 + 4 \cos 4x + \cos 8x) dx \\
&= \frac{1}{8} \left[\int 3 dx + 4 \int \cos 4x dx + \int \cos 8x dx \right] \\
&= \frac{1}{8} \left(3x + \frac{4 \sin 4x}{4} + \frac{\sin 8x}{8} \right) + C \\
&= \frac{3}{8} x + \frac{1}{8} \sin 4x + \frac{1}{64} \sin 8x + C
\end{aligned}$$

उत्तर

प्रश्न 12. $\frac{\sin^2 x}{1 + \cos x}$.

हल :

$$\begin{aligned}
\int \frac{\sin^2 x}{1 + \cos x} dx &= \int \frac{1 - \cos^2 x}{1 + \cos x} dx \\
&= \int \frac{(1 - \cos x)(1 + \cos x)}{(1 + \cos x)} dx \\
&= \int (1 - \cos x) dx = x - \sin x + C.
\end{aligned}$$

उत्तर

प्रश्न 13. $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$.

हल :

$$\begin{aligned}
\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx &= \int \frac{(2 \cos^2 x - 1) - (2 \cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx \\
&= \int \frac{2(\cos^2 x - \cos^2 \alpha)}{\cos x - \cos \alpha} dx \\
&= 2 \int \frac{(\cos x + \cos \alpha)(\cos x - \cos \alpha)}{\cos x - \cos \alpha} dx \\
&= 2 \int (\cos x + \cos \alpha) dx \\
&= 2 \left[\int \cos x dx + \cos \alpha \int 1 dx \right] \\
&= 2(\sin x + x \cos \alpha) + C.
\end{aligned}$$

उत्तर

प्रश्न 14. $\frac{\cos x - \sin x}{1 + \sin 2x}$.

हल : माना

$$I = \int \frac{\cos x - \sin x}{1 + \sin 2x} dx$$

∴
और

$$\begin{aligned}
 1 &= \cos^2 x + \sin^2 x \\
 \sin 2x &= 2 \cos x \sin x \\
 &= \int \frac{\cos x - \sin x}{\cos^2 x + \sin^2 x + 2 \cos x \sin x} dx \\
 &= \int \frac{\cos x - \sin x}{(\cos x + \sin x)^2} dx
 \end{aligned}$$

माना $\cos x + \sin x = t$ रखने पर
 $(-\sin x + \cos x) dx = dt$

$$\begin{aligned}
 \therefore I &= \int \frac{dt}{t^2} = \int t^{-2} dt = \frac{t^{-2+1}}{-2+1} + C \\
 &= -\frac{1}{t} + C = -\frac{1}{\cos x + \sin x} + C.
 \end{aligned}$$

उत्तर

प्रश्न 15. $\tan^3 2x \cdot \sec 2x$.

$$\begin{aligned}
 \text{हल : } \int \tan^3 2x \sec 2x dx &= \int \tan^2 2x \cdot \sec 2x \tan 2x dx \\
 &= \int (\sec^2 2x - 1) \sec 2x \tan 2x dx
 \end{aligned}$$

∴ $\sec 2x = t$ रखने पर
 $\therefore 2 \sec 2x \tan 2x dx = dt$

$$\text{या } \sec 2x \tan 2x = \frac{1}{2} dt$$

$$\begin{aligned}
 &= \frac{1}{2} \int (t^2 - 1) dt \\
 &= \frac{1}{2} \left(\frac{t^3}{3} - t \right) + C \\
 &= \frac{1}{2} \left(\frac{1}{3} \sec^3 2x - \sec 2x \right) + C \\
 &= \frac{1}{6} \sec^3 2x - \frac{1}{2} \sec 2x + C.
 \end{aligned}$$

उत्तर

प्रश्न 16. $\tan^4 x$.

$$\begin{aligned}
 \text{हल : } \int \tan^4 x dx &= \int \tan^2 x \cdot \tan^2 x dx \\
 &= \int \tan^2 x (\sec^2 x - 1) dx \\
 &= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx \\
 &= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1) dx
 \end{aligned}$$

∴ $\tan x = t$ रखने पर
 $\therefore$

$$\begin{aligned}
 \sec^2 x dx &= dt \\
 &= \int t^2 dt - \int 1 dt + \int 1 dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{t^3}{3} - t + x + C \\
&= \frac{1}{3} \tan^3 x - \tan x + x + C.
\end{aligned}$$

उत्तर

प्रश्न 17. $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$.

हल :

$$\begin{aligned}
\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx &= \int \frac{\sin^3 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^3 x}{\sin^2 x \cos^2 x} dx \\
&= \int \frac{\sin x}{\cos^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx \\
&= \int \sec x \tan x dx + \int \operatorname{cosec} x \cot x dx \\
&= \sec x - \operatorname{cosec} x + C.
\end{aligned}$$

उत्तर

प्रश्न 18. $\frac{\cos 2x + 2 \sin^2 x}{\cos^2 x}$.

हल :

$$\begin{aligned}
\int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} dx &= \int \frac{\cos^2 x - \sin^2 x + 2 \sin^2 x}{\cos^2 x} dx \quad [\because \cos 2x = \cos^2 x - \sin^2 x] \\
&= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x} dx \\
&= \int \frac{1}{\cos^2 x} dx = \int \sec^2 x dx \\
&= \tan x + C.
\end{aligned}$$

उत्तर

प्रश्न 19. $\frac{1}{\sin x \cos^3 x}$.

हल :

$$\begin{aligned}
\int \frac{1}{\sin x \cos^3 x} dx &= \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} dx \\
&= \int \left(\frac{\sin^2 x}{\sin x \cos^3 x} + \frac{\cos^2 x}{\sin x \cos^3 x} \right) dx \\
&= \int \left(\frac{\tan x}{\cos^2 x} + \frac{1}{\tan x \cdot \cos^2 x} \right) dx \\
&= \int \left(\tan x \sec^2 x + \frac{1}{\tan x} \sec^2 x \right) dx
\end{aligned}$$

अब $\tan x = t$ रखने पर

$$\sec^2 x dx = dt$$

$\therefore$

$$I = \int \left(t + \frac{1}{t} \right) dt = \frac{t^2}{2} + \log |x| + C$$

$$= \frac{1}{2} \tan^2 x + \log |\tan x| + C$$

$$= \log |\tan x| + \frac{1}{2} \tan^2 x + C.$$

उत्तर

प्रश्न 20. $\frac{\cos 2x}{(\cos x + \sin x)^2}.$

हल :
$$\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

$\because \cos x + \sin x = t$ रखने पर,
 $\therefore (-\sin x + \cos x) dx = dt$

$$= \int \frac{dt}{t} = \log |t| + C$$

$$= \log |\cos x + \sin x| + C.$$

उत्तर

प्रश्न 21. $\sin^{-1}(\cos x).$

हल :
$$\int \sin^{-1}(\cos x) dx = \int \sin^{-1}\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx$$

$$= \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2}x - \frac{x^2}{2} + C.$$

उत्तर

प्रश्न 22. $\frac{1}{\cos(x-a)\cos(x-b)}.$

हल :
$$\int \frac{1}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} dx$$

$$[\because \sin(A-B) = \sin A \cos B - \cos A \sin B]$$

$$= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)] dx$$

$$= \frac{1}{\sin(a-b)} [-\log |\cos(x-b)| + \log |\cos(x-a)|] + C$$

$$= \frac{1}{\sin(a-b)} \log \left| \frac{\cos(x-a)}{\cos(x-b)} \right| + C.$$

उत्तर

प्रश्न 23 एवं 24 में सही उत्तर का चयन कीजिए—

प्रश्न 23. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ बराबर है—

- (A) $\tan x + \cot x + C$ (B) $\tan x + \operatorname{cosec} x + C$
(C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

हल :
$$\begin{aligned}\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx &= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx - \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{dx}{\cos^2 x} - \int \frac{dx}{\sin^2 x} \\ &= \int \sec^2 x dx - \int \operatorname{cosec}^2 x dx \\ &= \tan x + \cot x + C.\end{aligned}$$

अतः विकल्प (A) सही है।

उत्तर

प्रश्न 24. $\int \frac{e^x(1+x)}{\cos^2(e^x)} dx$ बराबर है—

- (A) $-\cot(e^x) + C$ (B) $\tan(xe^x) + C$
(C) $\tan(e^x) + C$ (D) $\cot(e^x) + C$

हल :
$$\int \frac{e^x(1+x)}{\cos^2(e^x)} dx = \int \frac{(e^x + xe^x)dx}{\cos^2(e^x)}$$

मान लीजिए $xe^x = t$

$$\Rightarrow (e^x + xe^x)dx = dt$$

$$\begin{aligned}&= \int \frac{dt}{\cos^2 t} = \int \sec^2 t dt \\ &= \tan t + C \\ &= \tan(xe^x) + C\end{aligned}$$

अतः विकल्प (B) सही है।

उत्तर

प्रश्नावली 7.4

प्रश्न 1 से 23 तक के फलनों का समाकलन कीजिए—

प्रश्न 1. $\frac{3x^2}{x^6 + 1}$

हल :
$$\int \frac{3x^2}{x^6 + 1} dx$$

$\therefore x^3 = t$ रखने पर,

$\therefore$

$$3x^2 dx = dt$$

$$= \int \frac{dt}{t^2 + 1} = \tan^{-1} t + C = \tan^{-1} x^3 + C.$$

उत्तर

प्रश्न 2. $\frac{1}{\sqrt{1+4x^2}}$

हल :

यहाँ $2x = \tan \theta$ लेने पर,
 $\therefore 2dx = \sec^2 \theta d\theta$

$$\begin{aligned}
 \int \frac{1}{\sqrt{1+4x^2}} dx &= \int \frac{1}{\sqrt{1+(2x)^2}} dx \\
 &= \frac{1}{2} \int \frac{1}{\sqrt{1+\tan^2 \theta}} \sec^2 \theta d\theta \\
 &= \frac{1}{2} \int \frac{\sec^2 \theta}{\sec \theta} d\theta \\
 &= \frac{1}{2} \int \sec \theta d\theta = \frac{1}{2} \log |\sec \theta \tan \theta| + C \\
 &= \frac{1}{2} \log \left| \sqrt{1+4x^2} + 2x \right| + C \quad [\because \sec \theta = \sqrt{1+4x^2}] \\
 &= \frac{1}{2} \log \left| 2x + \sqrt{1+4x^2} \right| + C. \quad \text{उत्तर}
 \end{aligned}$$

प्रश्न 3. $\frac{1}{\sqrt{(2-x)^2+1}}$.

हल :

यहाँ $2-x = t$ रखने पर
 $\therefore -dx = dt$

$$\begin{aligned}
 \int \frac{1}{\sqrt{(2-x)^2+1}} dx &= -\int \frac{dx}{\sqrt{t^2+1}} = -\log |t + \sqrt{t^2+1}| + C \\
 &= -\log \left| (2-x) + \sqrt{(2-x)^2+1} \right| + C \\
 &= \log \left| \frac{1}{(2-x) + \sqrt{(2-x)^2+1}} \right| + C \\
 &= \log \left| \frac{1}{2-x + \sqrt{x^2-4x+5}} \right| + C. \quad \text{उत्तर}
 \end{aligned}$$

प्रश्न 4. $\frac{1}{\sqrt{9-25x^2}}$.

हल :

$$\begin{aligned}
 \int \frac{dx}{\sqrt{9-25x^2}} &= \frac{1}{5} \int \frac{dx}{\sqrt{\frac{9}{25}-x^2}} = \frac{1}{5} \int \frac{dx}{\sqrt{\left(\frac{3}{5}\right)^2-x^2}} \\
 &= \frac{1}{5} \sin^{-1} \left[\frac{x}{\frac{3}{5}} \right] + C = \frac{1}{5} \sin^{-1} \left(\frac{5x}{3} \right) + C. \quad \text{उत्तर}
 \end{aligned}$$

प्रश्न 5. $\frac{3x}{1+2x^4}$.

हल :

$\because x^2 = t$ रखने पर,
 $\therefore 2x dx = dt$

$$\int \frac{3x dx}{1+2x^4} = \int \frac{3x dx}{1+2(x^2)^2}$$

$$= \frac{3}{2} \int \frac{2x}{1+2(x^2)^2} dx = \frac{3}{2} \int \frac{dt}{1+2t^2}$$

$$= \frac{3}{2} \int \frac{dt}{\frac{1}{2} + t^2} = \frac{3}{2} \int \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2}$$

$$= \frac{3}{4} \sqrt{2} \tan^{-1} \left(\frac{t}{\frac{1}{\sqrt{2}}} \right) + C$$

$$\left[\because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$$

$$= \frac{3}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}t}{1} \right) + C$$

$$= \frac{3}{2\sqrt{2}} \tan^{-1} (\sqrt{2}x^2) + C.$$

उत्तर

प्रश्न 6. $\frac{x^2}{1-x^6}$.

हल :

$\because x^3 = t$ रखने पर,
 $\therefore 3x^2 dx = dt$

$$\int \frac{x^2}{1-x^6} dx = \int \frac{x^2 dx}{1-(x^3)^2}$$

$$= \frac{1}{3} \int \frac{dt}{1-t^2} = \frac{1}{3} \cdot \frac{1}{2} \log \frac{1+t}{1-t} + C$$

$$= \frac{1}{6} \log \left(\frac{1+x^3}{1-x^3} \right) + C.$$

उत्तर

प्रश्न 7. $\frac{x-1}{\sqrt{x^2-1}}$.

हल :

$$\begin{aligned} \int \frac{x-1}{\sqrt{x^2-1}} dx &= \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{dx}{\sqrt{x^2-1}} \\ &= \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx - \log(x + \sqrt{x^2-1}) \end{aligned}$$

$\because x^2 - 1 = t$ रखने पर
 $\therefore 2x dx = dt$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{dt}{\sqrt{t}} - \log(x + \sqrt{x^2 - 1}) \\
&= \frac{1}{2} \left[\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}} \right] - \log(x + \sqrt{x^2 - 1}) + C \\
&= \frac{1}{2} \left[\frac{1}{-\frac{1}{2}} \right] - \log(x + \sqrt{x^2 - 1}) + C \\
&= t^{\frac{1}{2}} - \log(x + \sqrt{x^2 - 1}) + C \\
&= \sqrt{x^2 - 1} - \log(x + \sqrt{x^2 - 1}) + C.
\end{aligned}$$

उत्तर

प्रश्न 8. $\frac{x^2}{\sqrt{x^6 + a^6}}.$

हल :

$\because x^3 = t$ रखने पर,
 $\therefore 3x^2 dx = dt$

$$\int \frac{x^2 dx}{\sqrt{x^6 + a^6}} = \int \frac{x^2 dx}{(x^3)^2 + a^6}$$

$$\begin{aligned}
&= \frac{1}{3} \int \frac{dt}{\sqrt{t^2 + a^6}} = \frac{1}{3} \int \frac{dt}{\sqrt{t^2 + (a^3)^2}} \\
&= \frac{1}{3} \log |t + \sqrt{t^2 + (a^3)^2}|
\end{aligned}$$

$$\left[\because \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log(x + \sqrt{x^2 + a^2}) \right]$$

$$= \frac{1}{3} \log |x^3 + \sqrt{x^6 + a^6}| + C.$$

उत्तर

प्रश्न 9. $\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}.$

हल :

यहाँ $\tan x = t$ रखने पर,
 $\therefore$

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx$$

$$\sec^2 x dx = dt$$

$$= \int \frac{dt}{\sqrt{t^2 + 4}} = \int \frac{dt}{\sqrt{t^2 + (2)^2}}$$

$$\left[\because \int \frac{dt}{\sqrt{x^2 + a^2}} dx = \log(x + \sqrt{x^2 + a^2}) \right]$$

$$= \log |t + \sqrt{t^2 + 4}| + C$$

$$= \log |\tan x + \sqrt{\tan^2 x + 4}| + C.$$

उत्तर

प्रश्न 10. $\frac{1}{\sqrt{x^2 + 2x + 2}}$.

हल :
$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + 1}} dx$$

$$= \log \left| (x+1) + \sqrt{(x+1)^2 + 1} \right| + C$$

$$\therefore = \log \left| (x+1) + \sqrt{x^2 + 2x + 2} \right| + C. \quad \text{उत्तर}$$

प्रश्न 11. $\frac{1}{9x^2 + 6x + 5}$.

हल :
$$\int \frac{1}{9x^2 + 6x + 5} dx = \frac{1}{9} \int \frac{dx}{x^2 + \frac{2}{3}x + \frac{5}{9}}$$

$$= \frac{1}{9} \int \frac{dx}{\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + \frac{5}{9} - \frac{1}{9}}$$

$$= \frac{1}{9} \int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \frac{4}{9}}$$

$$= \frac{1}{9} \int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$$

$$= \frac{1}{9} \cdot \frac{1}{\left(\frac{2}{3}\right)} \tan^{-1} \frac{x + \frac{1}{3}}{\frac{2}{3}} + C$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3x+1}{2} \right) + C. \quad \text{उत्तर}$$

प्रश्न 12. $\frac{1}{\sqrt{7-6x-x^2}}$.

हल :
$$\int \frac{1}{\sqrt{7-6x-x^2}} dx = \int \frac{dx}{\sqrt{7-(x^2+6x)}}$$

$$= \int \frac{dx}{\sqrt{7-(x^2+6x+9)+9}}$$

$$= \int \frac{dx}{\sqrt{16-(x+3)^2}} \left[\because \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} \right]$$

$$= \sin^{-1} \left(\frac{x+3}{4} \right) + C. \quad \text{उत्तर}$$

प्रश्न 13. $\frac{1}{\sqrt{(x-1)(x-2)}}$.

हल :

$$\begin{aligned} \int \frac{dx}{\sqrt{(x-1)(x-2)}} &= \int \frac{dx}{\sqrt{x^2 - 3x + 2}} \\ &= \int \frac{dx}{\sqrt{\left(x^2 - 3x + \frac{9}{4}\right) + 2 - \frac{9}{4}}} \\ &= \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}}} \\ &= \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \\ &= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}} \right| + C \\ &= \log \left| \frac{2x-3}{2} + \sqrt{x^2 - 3x + 2} \right| + C \\ &= \log \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + 2} \right| + C. \end{aligned}$$

उत्तर

प्रश्न 14. $\frac{1}{\sqrt{8+3x-x^2}}$.

हल :

$$\begin{aligned} \int \frac{dx}{\sqrt{8+3x-x^2}} &= \int \frac{dx}{\sqrt{8 - (x^2 - 3x)}} \\ &= \int \frac{dx}{\sqrt{8 - \left(x^2 - 3x + \frac{9}{4}\right) + \frac{9}{4}}} \\ &= \int \frac{dx}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2}\right)^2}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} \\ &\quad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \right] \\ &= \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}} \right) + C = \sin^{-1} \frac{2x-3}{\sqrt{41}} + C. \end{aligned}$$

उत्तर

प्रश्न 15. $\frac{1}{\sqrt{(x-a)(x-b)}}$.

हल :

$$\begin{aligned} \int \frac{dx}{\sqrt{(x-a)(x-b)}} &= \int \frac{dx}{\sqrt{x^2 - (a+b)x + ab}} \\ &= \int \frac{dx}{\sqrt{x^2 - (a+b)x + \left(\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab}} \\ &= \int \frac{dx}{\sqrt{\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}} \\ &= \int \frac{dx}{\sqrt{\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}} \\ &= \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log [(x + \sqrt{x^2 - a^2})] \\ &= \log \left(x - \frac{a+b}{2} + \sqrt{\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2} \right) + C \\ &= \log \left| x - \frac{a+b}{2} + \sqrt{x^2 - (a+b)x + ab} \right| + C \\ &= \log \left| x - \frac{a+b}{2} + \sqrt{(x-a)(x-b)} \right| + C. \quad \text{उत्तर} \end{aligned}$$

प्रश्न 16. $\frac{4x+1}{\sqrt{2x^2+x-3}}$.

हल :

$$\begin{aligned} \int \frac{4x+1}{\sqrt{2x^2+x-3}} dx \\ \because 2x^2+x-3 = t \text{ रखने पर,} \\ \therefore (4x+1) dx = dt \\ = \int \frac{dt}{\sqrt{t}} = \frac{t^{\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2\sqrt{t} + C \\ = 2\sqrt{2x^2+x-3} + C. \quad \text{उत्तर} \end{aligned}$$

प्रश्न 17. $\frac{x+2}{\sqrt{x^2-1}}$.

हल :

$$\begin{aligned} \int \frac{x+2}{\sqrt{x^2-1}} dx &= \int \frac{x}{\sqrt{x^2-1}} dx + \int \frac{2}{\sqrt{x^2-1}} dx \\ &= I_1 + I_2 \quad (\text{मान लीजिए}) \end{aligned}$$

$$I_1 = \frac{1}{2} \int \frac{2x \, dx}{\sqrt{x^2 - 1}}$$

I_1 में $x^2 - 1 = t$ रखने पर
 $2x \, dx = dt$

$$= \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \cdot \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= \sqrt{t} = \sqrt{x^2 - 1}$$

तथा

$$I_2 = 2 \int \frac{1}{\sqrt{x^2 - 1}} \, dx = 2 \log |x + \sqrt{x^2 - 1}|$$

$\therefore$

$$I_1 + I_2 = \sqrt{x^2 - 1} + 2 \log |x + \sqrt{x^2 - 1}| + C.$$

उत्तर

प्रश्न 18. $\frac{5x - 2}{1 + 2x + 3x^2}$

हल :

$$\int \frac{5x - 2}{1 + 2x + 3x^2} \, dx$$

अब

$$\begin{aligned} 5x - 2 &= A \frac{d}{dx} (1 + 2x + 3x^2) + B \\ &= A(2 + 6x) + B = 6Ax + (2A + B) \end{aligned}$$

x तथा अचर संख्याओं की तुलना करने पर

$$5 = 6A \text{ या } A = \frac{5}{6}$$

$$2A + B = -2 \text{ या } B = -2 - 2A = -2 - 2 \times \frac{5}{6}$$

$$B = -2 - \frac{5}{3} = -\frac{11}{3}$$

$$= \int \frac{\frac{5}{6}(2 + 6x) - \frac{11}{3}}{1 + 2x + 3x^2} \, dx$$

$$= \frac{5}{6} \int \frac{2 + 6x}{1 + 2x + 3x^2} \, dx - \frac{11}{3} \int \frac{dx}{1 + 2x + 3x^2}$$

$$= I_1 - \frac{11}{3} I_2 \quad (\text{मान लीजिए}) \quad \dots(i)$$

$\therefore$

$$I_1 = \frac{5}{6} \int \frac{2 + 6x}{1 + 2x + 3x^2} \, dx \quad \text{माना } 1 + 2x + 3x^2 = t$$

$$\therefore (2 + 6x) \, dx = dt$$

$$= \frac{5}{6} \int \frac{dt}{t} = \frac{5}{6} \log |t|$$

$$= \frac{5}{6} \log |1 + 2x + 3x^2| \quad \dots(ii)$$

तथा

$$\begin{aligned}
 I_2 &= \int \frac{dx}{1+2x+3x^2} \\
 &= \frac{1}{3} \int \frac{dx}{\frac{1}{3} + \frac{2}{3}x + x^2} \\
 &= \frac{1}{3} \int \frac{dx}{\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + \frac{1}{3} - \frac{1}{9}} \\
 &= \frac{1}{3} \int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}} \\
 &= \frac{1}{3} \cdot \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x + \frac{1}{3}}{\frac{\sqrt{2}}{3}} \right) \\
 &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \quad \dots(iii)
 \end{aligned}$$

समी. (ii) और (iii) का मान समी. (i) में रखने पर

$$\begin{aligned}
 &= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3} \times \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C \\
 &= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C. \quad \text{उत्तर}
 \end{aligned}$$

प्रश्न 19. $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$.

हल : $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx = \int \frac{6x+7}{\sqrt{x^2-9x+20}} dx$

अब $6x+7 = A \frac{d}{dx}(x^2-9x+20) + B = A(2x-9) + B$

 x के गुणांक तथा अचर संख्याओं की तुलना करने पर

$$6 = 2A \text{ या } A = 3$$

$$7 = -9A + B$$

$$= -27 + B$$

$$[\because A = 3]$$

 $\therefore$

$$B = 27 + 7 = 34$$

$$= 3 \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx + 34 \int \frac{1}{\sqrt{x^2-9x+20}} dx$$

$$= 3I_1 + 34I_2 \text{ (मान लीजिए)} \quad \dots(i)$$

अब

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx$$

 $\therefore x^2 - 9x + 20 = t$ रखने पर, $\therefore (2x-9) dx = dt$

$$\begin{aligned}
 &= \int \frac{dt}{\sqrt{t}} = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{t} \\
 &= 2\sqrt{x^2 - 9x + 20} \quad \dots(ii)
 \end{aligned}$$

तथा

$$\begin{aligned}
 I_2 &= \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx \\
 &= \frac{dx}{\sqrt{x^2 - 9x + \frac{81}{4} + 20 - \frac{81}{4}}} \\
 &= \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}} \\
 &\quad \left[\because \frac{1}{\sqrt{x^2 - a^2}} dx = \log \left| x + \sqrt{x^2 - a^2} \right| \right] \\
 &= \log \left| x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}} \right| \\
 &= \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| \quad \dots(iii)
 \end{aligned}$$

समी. (ii) व (iii) का मान समी. (i) में रखने पर

$$\begin{aligned}
 \therefore 3I_1 + 34I_2 &= 3 \times 2\sqrt{x^2 - 9x + 20} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + C \\
 &= 6\sqrt{x^2 - 9x + 20} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + C. \quad \text{उत्तर}
 \end{aligned}$$

प्रश्न 20. $\frac{x+2}{\sqrt{4x-x^2}}$

$$\begin{aligned}
 \text{हल : } \int \frac{x+2}{\sqrt{4x-x^2}} dx &= \int \frac{x+2}{\sqrt{-(x^2-4x+4)+4}} dx \\
 &= \int \frac{x+2}{\sqrt{4-(x-2)^2}} dx = \int \frac{x-2+4}{\sqrt{4-(x-2)^2}} dx \\
 &= \int \frac{x-2}{\sqrt{4-(x-2)^2}} dx + 4 \int \frac{1}{\sqrt{4-(x-2)^2}} dx
 \end{aligned}$$

$$\begin{aligned}
 \because 4 - (x-2)^2 &= t \text{ रखने पर} \\
 \therefore -2(x-2) dx &= dt
 \end{aligned}$$

$$= -\frac{1}{2} \int \frac{dt}{\sqrt{t}} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) \quad \left[\because \int \frac{dt}{a^2 - x^2} = \sin^{-1} \frac{x}{a} \right]$$

$$\begin{aligned}
&= -\frac{1}{2} \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + C \\
&= -\sqrt{t} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + C \\
&= -\sqrt{4-(x-2)^2} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + C \\
&= -\sqrt{4x-x^2} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + C.
\end{aligned}$$

उत्तर

प्रश्न 21. $\frac{x+2}{\sqrt{x^2+2x+3}}$.

हल :
$$\begin{aligned}
\int \frac{x+2}{\sqrt{x^2+2x+3}} dx &= \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}} dx \\
&= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{dx}{\sqrt{x^2+2x+3}}
\end{aligned}$$

अब प्रथम समाकलन में $x^2+2x+3=t$ रखने पर

$\therefore (2x+2) dx = dt$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{dt}{\sqrt{t}} + \int \frac{dx}{(x+1)^2+2} \\
&= \frac{1}{2} \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \log \left| (x+1) + \sqrt{(x+1)^2+2} \right| + C
\end{aligned}$$

$$= \sqrt{t} + \log \left| (x+1) + \sqrt{x^2+2x+3} \right| + C$$

$$= \sqrt{x^2+2x+3} + \log \left| (x+1) + \sqrt{x^2+2x+3} \right| + C.$$

उत्तर

प्रश्न 22. $\frac{x+3}{x^2-2x-5}$.

हल : $\int \frac{x+3}{x^2-2x-5} dx$

अब
$$\begin{aligned}
x+3 &= A \frac{d}{dx} (x^2-2x-5) + B \\
&= A(2x-2) + B
\end{aligned}$$

x तथा अचर संख्याओं की तुलना करने पर,

$$1 = 2A \text{ या } A = \frac{1}{2}$$

$$3 = -2A + B, B = 2A + 3 = 1 + 3 = 4$$

$$= \int \frac{\frac{1}{2}(2x-2) + 4}{x^2 - 2x - 5} dx$$

$$= \frac{1}{2} \int \frac{2x-2}{x^2 - 2x - 5} dx + 4 \int \frac{1}{x^2 - 2x - 5} dx$$

पहले समाकलन में $x^2 - 2x - 5 = t$ रखने पर

$$\therefore (2x-2) dx = dt$$

$$= \frac{1}{2} \int \frac{dt}{t} + 4 \int \frac{1}{(x-1)^2 - 6} dx$$

$$= \frac{1}{2} \log |t| + 4 \cdot \frac{1}{2\sqrt{6}} \log \frac{(x-1) - \sqrt{6}}{x-1 + \sqrt{6}} + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) \right]$$

$$= \frac{1}{2} \log |x^2 - 2x - 5| + \frac{2}{\sqrt{6}} \log \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} + C. \text{ उत्तर}$$

प्रश्न 23. $\frac{5x+3}{\sqrt{x^2+4x+10}}$

हल : $\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx$

$$5x+3 = A \frac{d}{dx} (x^2+4x+10) + B$$

$$= A(2x+4) + B$$

x तथा अचर संख्याओं के दोनों पक्षों की तुलना करने पर

$$5 = 2A \quad \text{या } A = \frac{5}{2}$$

$$3 = 4A + B \quad \text{या } B = 3 - 4A = 3 - 10 = -7$$

$$= \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2+4x+10}} dx$$

$$= \frac{5}{2} \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx - 7 \int \frac{dx}{\sqrt{x^2+4x+10}}$$

प्रथम समाकलन में $x^2 + 4x + 10 = t$ रखने पर तथा $(2x+4) dx = dt$

$$I = \frac{5}{2} \int \frac{dt}{\sqrt{t}} - 7 \int \frac{dx}{\sqrt{(x+2)^2 + 6}}$$

$$\begin{aligned}
&= \frac{5}{2} \cdot t^{\frac{-1}{2}+1} - 7 \log \left| (x+2) + \sqrt{(x+2)^2 + 6} \right| + C \\
&= 5\sqrt{t} - 7 \log \left| (x+2) + \sqrt{x^2 + 4x + 10} \right| + C \\
&= 5\sqrt{x^2 + 4x + 10} - 7 \log \left| (x+2) + \sqrt{x^2 + 4x + 10} \right| + C.
\end{aligned}$$

उत्तर

प्रश्न 24 व 25 में सही उत्तर का चयन कीजिए—

प्रश्न 24. $\int \frac{dx}{x^2 + 2x + 2}$ बराबर है—

- (A) $x \tan^{-1}(x+1) + C$ (B) $\tan^{-1}(x+1) + C$
 (C) $(x+1) \tan^{-1} x + C$ (D) $\tan^{-1} x + C$

हल :

$$\begin{aligned}
\int \frac{dx}{x^2 + 2x + 2} &= \int \frac{dx}{(x+1)^2 + 1} \\
&= \int \frac{dx}{(x+1)^2 + (1)^2} \\
&= \tan^{-1}(x+1) + C
\end{aligned}$$

अतः विकल्प (B) सही है।

उत्तर

प्रश्न 25. $\int \frac{dx}{\sqrt{9x - 4x^2}}$ बराबर है—

- (A) $\frac{1}{9} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$ (B) $\frac{1}{2} \sin^{-1} \left(\frac{8x-9}{9} \right) + C$
 (C) $\frac{1}{3} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$ (D) $\frac{1}{2} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$

हल :

$$\begin{aligned}
\int \frac{dx}{\sqrt{9x - 4x^2}} &= \int \frac{dx}{\sqrt{-(4x^2 - 9x)}} \\
&= \int \frac{dx}{\sqrt{-\left(4x^2 - 9x + \frac{81}{16}\right) - \frac{81}{16}}} \\
&= \int \frac{dx}{\sqrt{\frac{81}{16} - \left(2x - \frac{9}{4}\right)^2}}
\end{aligned}$$

मान लीजिए

$$2x - \frac{9}{4} = t$$

तब

$$2dx = dt$$

$$= \frac{1}{2} \int \frac{dt}{\left(\frac{9}{4}\right)^2 - (t)^2} = \frac{1}{2} \sin^{-1} \frac{t}{9/4} + C$$

$$\begin{aligned}
&= \frac{1}{2} \sin^{-1} \frac{4t}{9} + C \\
&= \frac{1}{2} \sin^{-1} \frac{4\left(2x - \frac{9}{4}\right)}{9} + C \\
&= \frac{1}{2} \sin^{-1} \left(\frac{8x - 9}{9} \right) + C
\end{aligned}$$

अतः विकल्प (B) सही है।

उत्तर

प्रश्नावली 7.5

प्रश्न 1 से 21 तक के प्रश्नों में परिमेय फलनों का समाकलन कीजिए—

प्रश्न 1. $\frac{x}{(x+1)(x+2)}$

हल : मान लीजिए

$$\begin{aligned}
\frac{x}{(x+1)(x+2)} &= \frac{A}{x+1} + \frac{B}{x+2} \\
\therefore x &= A(x+2) + B(x+1) \\
x = -1 \text{ रखने पर,} & \quad -1 = A \cdot 1 \text{ अर्थात् } A = -1 \\
x = -2 \text{ रखने पर,} & \quad -2 = B(-2+1) \text{ अर्थात् } B = 2
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{x}{(x+1)(x+2)} &= -\frac{1}{x+1} + \frac{2}{x+2} \\
\int \frac{x}{(x+1)(x+2)} dx &= -\int \frac{1}{x+1} dx + 2 \int \frac{1}{x+2} dx \\
&= -\log |x+1| + 2 \log |x+2| + C \\
&= -\log |x+1| + \log (x+2)^2 \\
&= \log \frac{(x+2)^2}{|x+1|} + C.
\end{aligned}$$

उत्तर

प्रश्न 2. $\frac{1}{x^2 - 9}$

हल :

$$\begin{aligned}
\frac{1}{x^2 - 9} &= \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \\
1 &= A(x+3) + B(x-3)
\end{aligned}$$

$$x = 3 \text{ रखने पर,} \quad 1 = A \cdot 6 \quad \text{या } A = \frac{1}{6}$$

$$x = -3 \text{ रखने पर,} \quad 1 = B(-6) \quad \text{या } B = -\frac{1}{6}$$

$$\therefore \frac{1}{x^2 - 9} = \frac{1}{6(x-3)} - \frac{1}{6(x+3)}$$

$$\therefore \int \frac{dx}{x^2 - 9} = \frac{1}{6} \int \frac{dx}{x-3} - \frac{1}{6} \int \frac{1}{x+3} dx$$

$$= \frac{1}{6} \log |x-3| - \frac{1}{6} \log |x+3| + C$$

$$= \frac{1}{6} \log \left| \frac{x-3}{x+3} \right| + C.$$

उत्तर

प्रश्न 3. $\frac{3x-1}{(x-1)(x-2)(x-3)}.$

हल : $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$x=1$ रखने पर, $2 = A(-1)(-2)$ या $A=1$

$x=2$ रखने पर, $5 = B.1(-1)$ या $B=-5$

$x=3$ रखने पर, $8 = C.2.1$ या $C=4$

अब $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{4}{x-3}$

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \frac{1}{x-1} dx - 5 \int \frac{1}{x-2} dx + 4 \int \frac{1}{x-3} dx$$

$$= \log |x-1| - 5 \log |x-2| + 4 \log |x-3| + C. \text{ उत्तर}$$

प्रश्न 4. $\frac{x}{(x-1)(x-2)(x-3)}.$

हल : मान लीजिए $\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$x=1$ लेने पर $1 = A(-1)(-2)$ या $A = \frac{1}{2}$

$x=2$ लेने पर $2 = B.1(-1)$ या $B = -2$

$x=3$ लेने पर $3 = C.2.1$ या $2C=3$ या $C = \frac{3}{2}$

$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$

$$\int \frac{x}{(x-1)(x-2)(x-3)} dx = \frac{1}{2} \int \frac{1}{x-1} dx - 2 \int \frac{1}{x-2} dx + \frac{3}{2} \int \frac{1}{x-3} dx$$

$$= \frac{1}{2} \log |x-1| - 2 \log |x-2| + \frac{3}{2} \log |x-3| + C. \text{ उत्तर}$$

प्रश्न 5. $\frac{2x}{x^2+3x+2}.$

हल : मान लीजिए $\frac{2x}{x^2+3x+2} = \frac{2x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$\therefore 2x = A(x+2) + B(x+1)$

$x = -1$ लेने पर
 $x = -2$ लेने पर

$$\begin{aligned} -2 &= A(1) & \text{अर्थात् } A &= -2 \\ -4 &= B(-1) & \text{अर्थात् } B &= 4 \end{aligned}$$

$$\therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{x+1} + \frac{4}{x+2}$$

$$\begin{aligned} \int \frac{2x}{(x+1)(x+2)} dx &= -2 \int \frac{1}{x+1} dx + 4 \int \frac{1}{x+2} dx \\ &= -2 \log |x+1| + 4 \log |x+2| + C \\ &= 4 \log |x+2| - 2 \log |x+1| + C. \end{aligned}$$

उत्तर

प्रश्न 6. $\frac{1-x^2}{x(1-2x)}$

हल :

$$\frac{1-x^2}{x(1-2x)} = \frac{1-x^2}{-2x^2+x}$$

अब अंश को हर से भाग देने पर,

$$\text{अर्थात् } \frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1-\frac{1}{2}x}{x-2x^2} = \frac{1}{2} + \frac{2-x}{2x(1-2x)} \quad \dots(i)$$

मान लीजिए

$$\begin{aligned} \frac{2-x}{x(1-2x)} &= \left[\frac{A}{x} + \frac{B}{1-2x} \right] \\ 2-x &= A(1-2x) + Bx \\ 2 &= A \quad \text{अर्थात् } A = 2 \end{aligned}$$

 $x = 0$ लेने पर,

$$x = \frac{1}{2} \text{ लेने पर, } \quad 2 - \frac{1}{2} = 0 + B \times \frac{1}{2}$$

या

$$\frac{B}{2} = \frac{3}{2}$$

या

$$B = 3$$

अतः

$$\frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

समीकरण (i) से,

$$\begin{aligned} \frac{1-x^2}{x(1-2x)} &= \frac{1}{2} + \frac{2-x}{2x(1-2x)} \\ &= \frac{1}{2} + \frac{1}{2} \left[\frac{2}{x} + \frac{3}{1-2x} \right] \\ &= \frac{1}{2} + \frac{1}{x} + \frac{3}{2(1-2x)} \end{aligned}$$

 $\therefore$

$$\begin{aligned} \int \frac{1-x^2}{x(1-2x)} dx &= \frac{1}{2} \int dx + \int \frac{1}{x} dx + \frac{3}{2} \int \frac{1}{1-2x} dx \\ &= \frac{1}{2} x + \log |x| + \frac{3}{2} \left(\frac{1}{-2} \right) \log |1-2x| + C \\ &= \frac{1}{2} x + \log |x| - \frac{3}{4} \log |1-2x| + C. \end{aligned}$$

उत्तर

प्रश्न 7. $\frac{x}{(x^2+1)(x-1)}$.

हल : मान लीजिए $\frac{x}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$$\therefore \begin{aligned} x &= A(x^2+1) + (Bx+C)(x-1) \\ x &= A(x^2+1) + B(x^2-x) + C(x-1) \end{aligned}$$

$$x=1 \text{ रखने पर, } 1 = A \times 2 \text{ या } A = \frac{1}{2}$$

x^2 तथा x के गुणांक की तुलना करने पर,

$$0 = A + B \text{ या } B = -A = -\frac{1}{2}$$

$$\text{तथा } 1 = -B + C \text{ या } C = 1 + B = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\begin{aligned} \text{अतः } \frac{x}{(x^2+1)(x-1)} &= \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} \\ &= \frac{1}{2(x-1)} - \frac{1}{2} \cdot \frac{x-1}{x^2+1} \\ &= \frac{1}{2(x-1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)} \\ \int \frac{x \, dx}{(x^2+1)(x-1)} &= \frac{1}{2} \int \frac{1}{x-1} \, dx - \frac{1}{4} \int \frac{2x}{x^2+1} \, dx + \frac{1}{2} \int \frac{1}{x^2+1} \, dx \\ &= \frac{1}{2} \log |x-1| - \frac{1}{4} \log |x^2+1| + \frac{1}{2} \tan^{-1} x + C. \quad \text{उत्तर} \end{aligned}$$

प्रश्न 8. $\frac{x}{(x-1)^2(x+2)}$.

हल : मान लीजिए

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\therefore x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$x=1 \text{ रखने पर, } 1 = B \times 3 \text{ या } B = \frac{1}{3}$$

$$x=-2 \text{ रखने पर, } -2 = C(-3)^2 \text{ या } C = -\frac{2}{9}$$

x^2 के गुणांक की तुलना करने पर,

$$0 = A + C \text{ या } A = -C = \frac{2}{9}$$

$$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

या
$$\int \frac{x}{(x-1)^2(x+2)} dx = \frac{2}{9} \int \frac{dx}{x-1} + \frac{1}{3} \int \frac{1}{(x-1)^2} dx - \frac{2}{9} \int \frac{1}{x+2} dx$$

$$= \frac{2}{9} \log |x-1| + \frac{1}{3} \frac{(x-1)^{-2+1}}{-2+1} - \frac{2}{9} \log |x+2| + C$$

$$= \frac{2}{9} \log |x-1| - \frac{1}{3(x-1)} - \frac{2}{9} \log |x+2| + C$$

$$= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + C.$$

उत्तर

प्रश्न 9. $\frac{3x+5}{x^3-x^2-x+1}$.

हल : $\therefore \frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{x^2(x-1)-(x-1)} = \frac{3x+5}{(x-1)(x^2-1)}$

$$= \frac{3x+5}{(x-1)(x-1)(x+1)}$$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\therefore 3x+5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$$x=1 \text{ लेने पर, } 8 = C \times 2 \text{ या } C=4$$

$x=-1$ लेने पर, $2 = A \times 4$ या $A = \frac{1}{2}$

अब x^2 के गुणांक की तुलना करने पर

$$0 = A + B \text{ या } B = -A = -\frac{1}{2}$$

अतः
$$\frac{3x+5}{x^3-x^2-x+1} = \frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{4}{(x-1)^2}$$

या
$$\int \frac{3x+5}{x^3-x^2-x+1} dx = \frac{1}{2} \int \frac{1}{x+1} dx - \frac{1}{2} \int \frac{1}{x-1} dx + 4 \int \frac{1}{(x-1)^2} dx$$

$$= \frac{1}{2} \log |x+1| - \frac{1}{2} \log |x-1| + 4 \frac{(x-1)^{-2+1}}{-2+1} + C$$

$$= \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{x-1} + C.$$

उत्तर

प्रश्न 10. $\frac{2x-3}{(x^2-1)(2x+3)}$.

हल :
$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)}$$

मान लीजिए
$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$$

$$\therefore 2x - 3 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x^2-1)$$

$$x = 1 \text{ लेने पर, } -1 = A \times 2 \times 5 \text{ या } A = -\frac{1}{10}$$

$$x = -1 \text{ लेने पर, } -5 = B(-2)(1) \text{ या } B = \frac{5}{2}$$

$$x = -\frac{3}{2} \text{ लेने पर, } -6 = C\left(\frac{9}{4}-1\right) = C\left(\frac{5}{4}\right)$$

$$\therefore C = \frac{-6 \times 4}{5} = -\frac{24}{5}$$

$$\text{अतः } \frac{2x-3}{(x^2-1)(2x+3)} = -\frac{1}{10(x-1)} + \frac{5}{2(x+1)} - \frac{24}{5} \cdot \frac{1}{2x+3}$$

$$\begin{aligned} \text{या } \int \frac{2x-3}{(x^2-1)(2x+3)} dx &= -\frac{1}{10} \int \frac{1}{x-1} dx + \frac{5}{2} \int \frac{1}{x+1} dx - \frac{24}{5} \int \frac{1}{2x+3} dx \\ &= -\frac{1}{10} \log |x-1| + \frac{5}{2} \log |x+1| - \frac{24}{5} \cdot \frac{1}{2} \log |2x+3| + C \\ &= \frac{5}{2} \log |x+1| - \frac{1}{10} \log |x-1| - \frac{12}{5} \log |2x+3| + C. \quad \text{उत्तर} \end{aligned}$$

$$\text{प्रश्न 11. } \frac{5x}{(x+1)(x^2-4)}.$$

$$\text{हल : मान लीजिए } \frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x-2)(x+2)}$$

$$= \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$\therefore 5x = A(x-2)(x+2) + B(x+1)(x+2) + C(x+1)(x-2)$$

$$x = -1 \text{ रखने पर, } -5 = A(1)(-3) \text{ या } A = \frac{5}{3}$$

$$x = -2 \text{ रखने पर, } -10 = C(-1)(-2-2) \text{ या } C = -\frac{10}{4} = -\frac{5}{2}$$

$$x = 2 \text{ रखने पर, } 10 = B(3)(4) \text{ या } 12B = 10 \text{ या } B = \frac{10}{12} = \frac{5}{6}$$

$$\therefore \frac{5x}{(x+1)(x^2-4)} = \frac{5}{3(x+1)} + \frac{5}{6(x-2)} - \frac{5}{2(x+2)}$$

$$\begin{aligned} \text{या } \int \frac{5x}{(x+1)(x^2-4)} dx &= \frac{5}{3} \int \frac{1}{x+1} dx + \frac{5}{6} \int \frac{1}{x-2} dx - \frac{5}{2} \int \frac{1}{x+2} dx \\ &= \frac{5}{3} \log |x+1| + \frac{5}{6} \log |x-2| - \frac{5}{2} \log |x+2| + C. \quad \text{उत्तर} \end{aligned}$$

$$\text{प्रश्न 12. } \frac{x^3+x+1}{x^2-1}.$$

हल : $\because \frac{x^3+x+1}{x^2-1}$ में अंश की घात हर की घात से अधिक है इसलिए x^3+x+1 को x^2-1 से भाग देने पर

$$\therefore \frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1} = x + \frac{2x}{x^2-1} + \frac{1}{x^2-1}$$

$$\begin{aligned} \text{या} \quad \int \frac{x^3+x+1}{x^2-1} dx &= \int x dx + \int \frac{2x}{x^2-1} dx + \int \frac{1}{x^2-1} dx \\ &= \frac{x^2}{2} + \log |x^2-1| + \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + C \\ &= \frac{x^2}{2} + \log |(x-1)(x+1)| + \frac{1}{2} \log |x-1| - \frac{1}{2} \log |x+1| + C \\ &= \frac{x^2}{2} + \log |x-1| + \log |x+1| \\ &\quad + \frac{1}{2} \log |x-1| - \frac{1}{2} \log |x+1| + C \\ &= \frac{x^2}{2} + \frac{3}{2} \log |x-1| + \frac{1}{2} \log |x+1| + C. \end{aligned}$$

उत्तर

प्रश्न 13. $\frac{2}{(1-x)(1+x^2)}$.

$$\begin{aligned} \text{हल : मान लीजिए} \quad \frac{2}{(1-x)(1+x^2)} &= \frac{A}{1-x} + \frac{Bx+C}{1+x^2} \\ 2 &= A(1+x^2) + (Bx+C)(1-x) \\ &= A(1+x^2) + B(x-x^2) + C(1-x) \\ x=1 \text{ लेने पर,} \quad 2 &= A(2) \text{ या } 2A=2 \text{ या } A=1 \\ x^2 \text{ तथा } x \text{ के गुणांकों की तुलना करने पर} \\ 0 &= A-B \text{ या } B=A=1 \\ 0 &= B-C \text{ या } C=B=1 \end{aligned}$$

$$\begin{aligned} \therefore \frac{2}{(1-x)(1+x^2)} &= \frac{1}{1-x} + \frac{x+1}{1+x^2} \\ &= \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2} \\ \int \frac{2}{(1-x)(1+x^2)} dx &= \int \frac{1}{1-x} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx \\ &= -\int \frac{dx}{x-1} + \frac{1}{2} \int \frac{2x}{x^2+1} dx + \int \frac{2x}{x^2+1} \\ &= -\log |x-1| + \frac{1}{2} \log (1+x^2) + \tan^{-1} x + C. \end{aligned}$$

उत्तर

प्रश्न 14. $\frac{3x-1}{(x-2)^2}$.

हल : मान लीजिए

$$\frac{3x-1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$x = 2$ लेने पर,

$$3x-1 = A(x-2) + B$$

दोनों पक्षों के x के गुणांकों की तुलना करने पर

$$6-1 = B \text{ या } B = 5$$

$\therefore$

$$\frac{3x-1}{(x-2)^2} = \frac{3}{x-2} + \frac{5}{(x-2)^2}$$

या

$$\begin{aligned} \int \frac{3x-1}{(x-2)^2} dx &= 3 \int \frac{dx}{x-2} + 5 \int \frac{dx}{(x-2)^2} \\ &= 3 \log |x-2| - \frac{5}{x-2} + C. \end{aligned}$$

उत्तर

प्रश्न 15. $\frac{1}{x^4-1}$.

हल : मान लीजिए

$$\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + C(x^3-x) + D(x^2-1)$$

$x = 1$ लेने पर,

$$1 = A \times 2 \times 2 \text{ या } A = \frac{1}{4}$$

$x = -1$ रखने पर,

$$1 = B(-2)(2) \text{ या } -4B = 1 \text{ या } B = -\frac{1}{4}$$

x^3 तथा x^2 के गुणांकों की तुलना करने पर,

$$0 = A + B + C \text{ या } C = -A - B = -\frac{1}{4} + \frac{1}{4} = 0$$

$$0 = A - B + D \text{ या } D = B - A = -\frac{1}{4} - \frac{1}{4} = -\frac{2}{4} = -\frac{1}{2}$$

$\therefore$

$$\frac{1}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$$

या

$$\begin{aligned} \int \frac{1}{x^4-1} dx &= \frac{1}{4} \int \frac{1}{x-1} dx - \frac{1}{4} \int \frac{1}{x+1} dx - \frac{1}{2} \int \frac{1}{x^2+1} dx \\ &= \frac{1}{4} \log |x-1| - \frac{1}{4} \log |x+1| - \frac{1}{2} \tan^{-1} x + C \\ &= \frac{1}{4} \log \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + C. \end{aligned}$$

उत्तर

प्रश्न 16. $\frac{1}{x(x^n + 1)}$.

हल : मान लीजिए $\int \frac{1}{x(x^n + 1)} dx = \int \frac{x^{n-1}}{x^{n-1} \cdot x(x^n + 1)} dx = \int \frac{x^{n-1} dx}{x^n(x^n + 1)}$

यहाँ $x^n = t$ रखने पर $n x^{n-1} dx = dt$

$$= \frac{1}{n} \int \frac{dt}{t(t+1)}$$

अब $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$

$$\therefore 1 = A(t+1) + Bt$$

$t = 0$ लेने पर, $1 = A$ या $A = 1$

$t = -1$ लेने पर, $1 = B(-1)$ या $B = -1$

$$\therefore \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\frac{1}{n} \int \frac{dt}{t(t+1)} = \frac{1}{n} \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt$$

$$= \frac{1}{n} [\log |t| - \log |t+1|] + C$$

$$= \frac{1}{n} \log \left| \frac{t}{t+1} \right| + C$$

$$= \frac{1}{n} \log \left| \frac{x^n}{x^n + 1} \right| + C.$$

उत्तर

प्रश्न 17. $\frac{\cos x}{(1 - \sin x)(2 - \sin x)}$.

हल : $\int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} dx$

$\therefore \sin x = t$ रखने पर

$$\therefore \cos x dx = dt$$

$$= \int \frac{dt}{(1-t)(2-t)}$$

अब $\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t}$ (i)

$$\therefore 1 = A(2-t) + B(1-t)$$

$t = 1$ लेने पर, $1 = A \times 1$ या $A = 1$

$t = 2$ लेने पर, $1 = B(1-2)$ या $-B = 1$ या $B = -1$

A तथा B के मान समीकरण (i) में रखने पर

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{1-t} - \frac{1}{2-t}$$

$$\begin{aligned}
& \int \frac{dt}{(1-t)(2-t)} \\
&= \int \frac{1}{1-t} dt - \int \frac{1}{2-t} dt \\
&= -\log |1-t| + \log |2-t| + C \\
&= \log \left| \frac{2-t}{1-t} \right| + C \\
&= \log \left| \frac{2-\sin x}{1-\sin x} \right| + C.
\end{aligned}$$

उत्तर

प्रश्न 18. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$.

हल : $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ में $x^2 = t$ रखने पर

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)} = \frac{t^2+3t+2}{t^2+7t+12}$$

इस परिमेय में अंश व हर की घात बराबर है इसलिए अंश को हर से भाग देने पर

$$\begin{aligned}
\therefore \frac{t^2+3t+2}{t^2+7t+12} &= 1 - \frac{4t+10}{t^2+7t+12} \\
&= 1 - \frac{2(2t+5)}{t^2+7t+12} \quad \dots(i)
\end{aligned}$$

अब लीजिए $\frac{2t+5}{t^2+7t+12} = \frac{2t+5}{(t+3)(t+4)} = \frac{A}{t+3} + \frac{B}{t+4}$

$$\therefore 2t+5 = A(t+4) + B(t+3)$$

$$t = -3 \text{ लेने पर, } -1 = A \times 1 \text{ या } A = -1$$

$$t = -4 \text{ लेने पर, } -3 = B(-4+3) \text{ या } B = 3$$

$$\therefore \frac{2t+5}{t^2+7t+12} = -\frac{1}{t+3} + \frac{3}{t+4}$$

समीकरण (i) में $t = x^2$ रखने पर,

$$\int \left\{ 1 - 2 \left(-\frac{1}{x^2+3} + \frac{3}{x^2+4} \right) \right\} dx$$

$$= \int dx + 2 \int \frac{1}{x^2+3} dx - 6 \int \frac{1}{x^2+4} dx$$

$$= \int dx + 2 \int \frac{1}{x^2+(\sqrt{3})^2} dx - 6 \int \frac{1}{x^2+(2)^2} dx$$

$$\left[\because \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$$

$$\begin{aligned}
&= x + 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - \frac{6}{2} \tan^{-1} \frac{x}{2} + C \\
&= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \frac{x}{2} + C.
\end{aligned}$$

उत्तर

प्रश्न 19. $\frac{2x}{(x^2+1)(x^2+3)}$.

हल : $\int \frac{2x}{(x^2+1)(x^2+3)} dx$, मान लीजिए $x^2 = t$, $\therefore 2x dt = dt$

$$= \int \frac{dt}{(t+1)(t+3)}$$

अब

$$\begin{aligned}
\frac{1}{(t+1)(t+3)} &= \frac{A}{t+1} + \frac{B}{t+3} \\
1 &= A(t+3) + B(t+1)
\end{aligned}$$

 $t = -1$ लेने पर,

$$1 = A \times 2 \text{ या } A = \frac{1}{2}$$

 $t = -3$ लेने पर,

$$1 = B(-3+1) \text{ या } -2B = 1 \text{ या } B = -\frac{1}{2}$$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$$

$$\begin{aligned}
\therefore \int \frac{1}{(t+1)(t+3)} dt &= \frac{1}{2} \int \frac{1}{t+1} dt - \frac{1}{2} \int \frac{1}{t+3} dt \\
&= \frac{1}{2} \log |t+1| - \frac{1}{2} \log |t+3| + C \\
&= \frac{1}{2} \log \left| \frac{t+1}{t+3} \right| + C \\
&= \frac{1}{2} \log \left| \frac{x^2+1}{x^2+3} \right| + C.
\end{aligned}$$

उत्तर

प्रश्न 20. $\frac{1}{x(x^4-1)}$.

हल : $\int \frac{1}{x(x^4-1)} dx = \int \frac{x^3}{x^4(x^4-1)} dx$

 $\therefore x^4 = t$ रखने पर

$$\therefore 4x^3 dx = dt$$

$$= \frac{1}{4} \int \frac{dt}{t(t-1)}$$

....(i)

अब लीजिए

$$\begin{aligned}
\frac{1}{(t-1)t} &= \frac{A}{t-1} + \frac{B}{t} \\
1 &= At + B(t-1)
\end{aligned}$$

$t = 0$ लेने पर,

$$1 = B(-1) \text{ या } B = -1$$

$t = 1$ लेने पर,

$$1 = A \times 1 \text{ या } A = 1$$

$\therefore$

$$\frac{1}{(t-1)t} = \frac{1}{t-1} - \frac{1}{t}$$

अब समीकरण (i) से,

$$\begin{aligned} &= \frac{1}{4} \int \left(\frac{1}{t-1} - \frac{1}{t} \right) dt \\ &= \frac{1}{4} \int \frac{1}{t-1} dt - \frac{1}{4} \int \frac{1}{t} dt \\ &= \frac{1}{4} \log |t-1| - \frac{1}{4} \log |t| + C \\ &= \frac{1}{4} \log \left| \frac{t-1}{t} \right| + C \\ &= \frac{1}{4} \log \left| \frac{x^4-1}{x^4} \right| + C. \end{aligned}$$

उत्तर

प्रश्न 21. $\frac{1}{(e^x-1)}$

हल : $\int \frac{dx}{(e^x-1)} = \int \frac{e^x dx}{e^x(e^x-1)}$

मान लीजिए $e^x = t$ हो, तब $e^x dx = dt$

$$= \int \frac{dt}{t(t-1)}$$

अब लीजिए

$$\frac{1}{t(t-1)} = \frac{A}{t-1} + \frac{B}{t}$$

$$1 = At + B(t-1)$$

$t = 1$ लेने पर,

$$1 = A \times 1 \text{ या } A = 1$$

$t = 0$ लेने पर,

$$1 = B(-1) \text{ या } B = -1$$

$\therefore$

$$\frac{1}{t(t-1)} = \frac{1}{t-1} - \frac{1}{t}$$

या

$$\begin{aligned} &\int \frac{1}{t(t-1)} dt \\ &= \int \frac{1}{t-1} dt - \int \frac{1}{t} dt \\ &= \log |t-1| - \log |t| + C \\ &= \log \left| \frac{t-1}{t} \right| + C \\ &= \log \left| \frac{e^x-1}{e^x} \right| + C. \end{aligned}$$

($t = e^x$ रखने पर) उत्तर

प्रश्न 22 व 23 में सही उत्तर का चयन कीजिए—

प्रश्न 22. $\int \frac{x \, dx}{(x-1)(x-2)}$ बराबर है—

(A) $\log \left| \frac{(x-1)^2}{x-2} \right| + C$

(B) $\log \left| \frac{(x-2)^2}{x-1} \right| + C$

(C) $\log \left| \left(\frac{x-1}{x-2} \right)^2 \right| + C$

(D) $\log |(x-1)(x-2)| + C$

हल :

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

या

$$x = A(x-2) + B(x-1)$$

$x = 1$ लेने पर,

$$1 = A(1-2)$$

या

$$-A = 1 \text{ या } A = -1$$

$x = 2$ लेने पर,

$$2 = B(2-1)$$

या

$$B = 2$$

अब

$$\frac{x}{(x-1)(x-2)} = -\frac{1}{x-1} + \frac{2}{x-2}$$

$\therefore$

$$\begin{aligned} \int \frac{x \, dx}{(x-1)(x-2)} &= -\int \frac{1}{x-1} \, dx + 2 \int \frac{1}{x-2} \, dx \\ &= -\log |x-1| + 2 \log |x-2| + C \\ &= \log \left| \frac{(x-2)^2}{x-1} \right| + C. \end{aligned}$$

अतः विकल्प (B) सही है।

उत्तर

प्रश्न 23. $\int \frac{dx}{x(x^2+1)}$ बराबर है—

(A) $\log |x| - \frac{1}{2} \log (x^2+1) + C$

(B) $\log |x| + \frac{1}{2} \log (x^2+1) + C$

(C) $-\log |x| + \frac{1}{2} \log (x^2+1) + C$

(D) $\frac{1}{2} \log |x| + \log (x^2+1) + C$

हल :

$$\int \frac{dx}{x(x^2+1)} = \int \frac{x \, dx}{x^2(x^2+1)}$$

मान लीजिए $x^2 = t$ हो, तब $2x \, dx = dt$

$$= \frac{1}{2} \int \frac{dt}{t(t+1)}$$

अब

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

$$1 = A(t+1) + Bt$$

 $t = 0$ लेने पर,

$$1 = A \text{ या } A = 1$$

 $t = -1$ लेने पर,

$$1 = B \times (-1) \text{ या } B = -1$$

 $\therefore$

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

अतः

$$\frac{1}{2} \int \frac{dt}{t(t+1)} = \frac{1}{2} \left[\int \frac{1}{t} dt - \int \frac{1}{t+1} dt \right]$$

$$= \frac{1}{2} \log |t| - \frac{1}{2} \log |t+1| + C$$

$$= \frac{1}{2} \log |x^2| - \frac{1}{2} \log |x^2+1| + C \quad (t = x^2 \text{ रखने पर})$$

$$= \log |x| - \frac{1}{2} \log |x^2+1| + C.$$

अतः विकल्प (A) सही है।

उत्तर

प्रश्नावली 7.6

प्रश्न 1 से 22 तक के प्रश्नों के फलनों का समाकलन कीजिए—

प्रश्न 1. $x \sin x$.हल : $\int x \sin x \, dx$

$$\int uv \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$$

 $u = x$ तथा $v = \sin x$ लेने पर,

$$= x \int \sin x \, dx - \int \left[\left(\frac{d}{dx} x \right) \int \sin x \, dx \right] dx$$

$$= x(-\cos x) - \int 1(-\cos x) \, dx$$

$$= -\cos x + \sin x + C.$$

उत्तर

प्रश्न 2. $x \sin 3x$.हल : $\int x \sin 3x \, dx$

खण्डशः समाकलन से,

$$\int uv \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$$

 $u = x$ तथा $v = 3x$ लेने पर

$$\begin{aligned}
\int x \cdot \sin 3x \, dx &= x \left(-\frac{\cos 3x}{3} \right) - \int 1 \cdot \left(-\frac{\cos 3x}{3} \right) dx \\
&= -\frac{x \cos 3x}{3} + \frac{1}{3} \int \cos 3x \, dx \\
&= -\frac{x \cos 3x}{3} + \frac{1}{9} \sin 3x + C \\
&= -\frac{x}{3} \cos 3x + \frac{1}{9} \sin 3x + C.
\end{aligned}$$

उत्तर

i zu 3. $x^2 e^x$.हल : $\int x^2 e^x \, dx$

खण्डशः समाकलन से,

$$\int uv \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$$

 $u = x^2$ तथा $v = e^x$ लेने पर

$$\begin{aligned}
&= x^2 \int e^x \, dx - \int \left(\frac{d}{dx} x^2 \int e^x \, dx \right) \\
&= x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \int x e^x \, dx
\end{aligned}$$

पुनः खण्डशः समाकलन करने पर

$$\begin{aligned}
&= x^2 e^x - 2 \left[x \int e^x \, dx - \int (1 \cdot \int e^x \, dx) dx \right] \\
&= x^2 e^x - 2 \left[x e^x - \int e^x \, dx \right] \\
&= x^2 e^x - 2x e^x + 2e^x + C \\
&= e^x (x^2 - 2x + 2) + C.
\end{aligned}$$

उत्तर

प्रश्न 4. $x \log x$.हल : $\int x \log x \, dx$

खण्डशः समाकलन से,

$$\int uv \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$$

 $u = \log x$ तथा $v = x$ रखने पर,

$$\int x \log x \, dx$$

$$\begin{aligned}
&= (\log x) \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \\
&= \frac{x^2}{2} \log x - \frac{1}{2} \int x \, dx \\
&= \frac{x^2}{2} \log x - \frac{1}{2} \cdot \frac{x^2}{2} + C \\
&= \frac{x^2}{2} \log x - \frac{1}{4} x^2 + C.
\end{aligned}$$

उत्तर

प्रश्न 5. $x \log 2x$.

हल : $\int x \log 2x dx$

खण्डशः समाकलन से,

$$\int uv dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx$$

यहाँ $u = \log 2x$ तथा $v = x$ लेने पर,

$$\begin{aligned} &= \log 2x \cdot \int x dx - \int \left[\frac{d}{dx} (\log 2x) \int x dx \right] dx \\ &= (\log 2x) \times \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \\ &= \frac{x^2}{2} \log 2x - \frac{1}{2} \int x dx \\ &= \frac{x^2}{2} \log 2x - \frac{x^2}{4} + C. \end{aligned}$$

उत्तर

प्रश्न 6. $x^2 \log x$.

हल : $\int x^2 \log x dx = \int (\log x) \cdot x^2 dx$

खण्डशः समाकलन से,

$$\int uv dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx$$

यहाँ $u = \log x$ तथा $v = x^2$ लेने पर

$$\begin{aligned} &= (\log x) \int x^2 dx - \int \left[\frac{d}{dx} (\log x) \cdot \int x^2 dx \right] dx \\ &= (\log x) \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx \\ &= \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 dx \\ &= \frac{x^3}{3} \log x - \frac{x^3}{9} + C. \end{aligned}$$

उत्तर

प्रश्न 7. $x \sin^{-1} x$.

हल : $\int x \sin^{-1} x dx = \int (\sin^{-1} x) x dx$

खण्डशः समाकलन से,

$$\int uv dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx$$

यहाँ $u = \sin^{-1} x$ तथा $v = x$ लेने पर

$$= (\sin^{-1} x) \int x dx - \int \left[\frac{d}{dx} (\sin^{-1} x) \cdot \int x dx \right] dx$$

$$\begin{aligned}
&= (\sin^{-1} x) \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx \\
&= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx \\
&= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx \\
&= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx \\
&= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right] - \frac{1}{2} \sin^{-1} x + C \\
&\quad \left[\because \int \sqrt{a^2-x^2} = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right] \\
&= \frac{x^2}{2} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + C \\
&= \frac{1}{4} (2x^2 - 1) \sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} + C.
\end{aligned}$$

उत्तर

प्रश्न 8. $x \tan^{-1} x$.

हल : $\int x \tan^{-1} x dx = \int (\tan^{-1} x) \cdot x dx$

खण्डशः समाकलन करने पर,

$$\begin{aligned}
&= (\tan^{-1} x) \int x dx - \int \left(\frac{d}{dx} \tan^{-1} x \int x dx \right) dx \\
&= (\tan^{-1} x) \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{x^2+1} \right) dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{1+x^2} dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C.
\end{aligned}$$

उत्तर

प्रश्न 9. $x \cos^{-1} x$.

हल : $\int x \cos^{-1} x dx = \int (\cos^{-1} x) \cdot x dx$

खण्डशः समाकलन करने पर,

$$\begin{aligned}
&= (\cos^{-1} x) \int x dx - \int \left[\frac{d}{dx} (\cos^{-1} x) \int x dx \right] dx \\
&= \cos^{-1} x \cdot \frac{x^2}{2} - \int -\frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx + C
\end{aligned}$$

मान लीजिए $x = \cos \theta$
तब $dx = -\sin \theta d\theta$

$$\begin{aligned}
&= \frac{x^2}{2} \cos^{-1} x + \frac{1}{2} \int \frac{\cos^2 \theta}{\sqrt{1-\cos^2 \theta}} [-\sin \theta] d\theta \\
&= \frac{x^2}{2} \cos^{-1} x - \frac{1}{2} \int \frac{\cos^2 \theta}{\sin \theta} \times \sin \theta d\theta \\
&= \frac{x^2}{2} \cos^{-1} x - \frac{1}{4} \int (1 + \cos 2\theta) d\theta \\
&= \frac{x^2}{2} \cos^{-1} x - \frac{1}{4} \int 1 d\theta - \frac{1}{4} \int \cos 2\theta d\theta \\
&= \frac{x^2}{2} \cos^{-1} x - \frac{\theta}{4} - \frac{1}{4} \cdot \frac{\sin 2\theta}{2} + C \\
&= \frac{x^2}{2} \cos^{-1} x - \frac{\theta}{4} - \frac{1}{8} \cdot 2 \sin \theta \cos \theta + C \\
&= \frac{x^2}{2} \cos^{-1} x - \frac{\cos^{-1} x}{4} - \frac{\sqrt{1-x^2}}{4} \cdot x + C \\
&= \frac{x^2}{2} \cos^{-1} x - \frac{1}{4} \cos^{-1} x - \frac{x\sqrt{1-x^2}}{4} + C \\
&= \frac{(2x^2-1)}{4} \cos^{-1} x - \frac{x\sqrt{1-x^2}}{4} + C \\
&= (2x^2-1) \frac{\cos^{-1} x}{4} - \frac{x}{4} \sqrt{1-x^2} + C.
\end{aligned}$$

उत्तर

प्रश्न 10 $(\sin^{-1} x)^2$.

हल : $\int (\sin^{-1} x)^2 dx$

मान लीजिए $\sin^{-1} x = \theta$ हो, तब $x = \sin \theta$, $dx = \cos \theta d\theta$

$$= \int \theta^2 \cos \theta d\theta$$

खण्डशः समाकलन करने पर,

$$= \theta^2 \sin \theta - \int 2\theta \sin \theta d\theta$$

पुनः खण्डशः समाकलन करने पर

$$\begin{aligned}
&= \theta^2 \sin \theta - 2[\theta(-\cos \theta) - \int 1(-\cos \theta) d\theta] \\
&= \theta^2 \sin \theta + 2\theta \cos \theta - 2 \int \cos \theta d\theta
\end{aligned}$$

$\sin \theta = x$ या $\theta = \sin^{-1} x$ रखने पर

$$= \theta^2 \sin \theta + 2\theta \cos \theta - 2 \sin \theta + C$$

$$= \theta^2 \sin \theta + 2\theta \sqrt{1 - \sin^2 \theta} - 2 \sin \theta + C$$

$$= x (\sin^{-1} x)^2 + 2\sqrt{1 - x^2} (\sin^{-1} x) - 2x + C$$

$$= (\sin^{-1} x)^2 x + 2\sqrt{1 - x^2} \sin^{-1} x - 2x + C.$$

उत्तर

प्रश्न 11. $\frac{x \cos^{-1} x}{\sqrt{1 - x^2}}.$

हल :

$$\frac{x \cos^{-1} x}{\sqrt{1 - x^2}}$$

मान लीजिए $\cos^{-1} x = t$ हो, तब, $x = \cos t$, $-\frac{1}{\sqrt{1 - x^2}} dx = dt$

$$\begin{aligned} \therefore \int \frac{x \cos^{-1} x}{\sqrt{1 - x^2}} dx &= -\int \cot t \cdot t \cdot dt \\ &= -[t(\sin t) - \int 1 \cdot \sin t \cdot dt] \\ &= -t \sin t + (-\cos t) + C \\ &= -\sqrt{1 - \cos^2 t} \cdot t - \cos t + C \\ &= -\sqrt{1 - x^2} \cos^{-1} x - x + C \\ &= -[\sqrt{1 - x^2} \cos^{-1} x + x] + C. \end{aligned}$$

उत्तर

प्रश्न 12. $x \sec^2 x.$

हल : $\int x \sec^2 x \cdot dx$

खण्डशः समाकलन करने पर,

$$\begin{aligned} &= x \tan x - \int 1 \cdot \tan x \cdot dx \\ &= x \tan x + \log |\cos x| + C. \end{aligned}$$

उत्तर

प्रश्न 13. $\tan^{-1} x.$

हल : $\int \tan^{-1} x \cdot dx = \int (\tan^{-1} x) \cdot 1 \cdot dx$

खण्डशः समाकलन करने पर,

$$\begin{aligned} &= (\tan^{-1} x) \cdot x - \int \frac{1}{1 + x^2} \cdot x \cdot dx \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1 + x^2} dx, \quad 1 + x^2 = t \text{ रखने पर} \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{dt}{t} \quad \Rightarrow 2dx = dt \\ &= x \tan^{-1} x - \frac{1}{2} \log |t| + C \\ &= x \tan^{-1} x - \frac{1}{2} \log |1 + x^2| + C. \end{aligned}$$

उत्तर

प्रश्न 14. $x (\log x)^2$.

हल : $\int x(\log x)^2 dx = \int (\log x)^2 \cdot x dx$

खण्डशः समाकलन करने पर,

$$\begin{aligned} &= (\log x)^2 \cdot \frac{x^2}{2} - \int 2(\log x) \frac{1}{x} \cdot \frac{x^2}{2} dx \\ &= \frac{x^2}{2} (\log x)^2 - \int (\log x) \cdot x dx \end{aligned}$$

पुनः खण्डशः समाकलन करने पर,

$$\begin{aligned} &= \frac{x^2}{2} (\log x)^2 - \log(x) \frac{x^2}{2} + \int \frac{1}{x} \cdot \frac{x^2}{2} dx \\ &= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int x dx \\ &= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \cdot \frac{x^2}{2} + C \\ &= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} (\log x) + \frac{x^2}{4} + C. \end{aligned}$$

उत्तर

प्रश्न 15. $(x^2 + 1) \log x$.

हल : $\int (x^2 + 1) \log x dx = \int (\log x)(x^2 + 1) dx$

खण्डशः समाकलन करने पर,

$$\begin{aligned} &= (\log x) \cdot \left(\frac{x^3}{3} + x \right) - \int \frac{1}{x} \left(\frac{x^3}{3} + x \right) dx \\ &= \left(\frac{x^3}{3} + x \right) \log x - \int \left(\frac{x^2}{3} + 1 \right) dx \\ &= \left(\frac{x^3}{3} + x \right) \log x - \frac{1}{3} \int x^2 dx - \int 1 dx \\ &= \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C. \end{aligned}$$

उत्तर

प्रश्न 16. $e^x (\sin x + \cos x)$.

हल : $\int e^x (\sin x + \cos x) dx$

$$\begin{aligned} &= \int e^x \sin x dx + \int e^x \cos x dx \\ &= \sin x \cdot e^x - \int \cos x \cdot e^x dx + \int e^x \cos x dx \\ &= \sin x \cdot e^x + C. \end{aligned}$$

उत्तर

प्रश्न 17. $\frac{xe^x}{(1+x)^2}$.

हल :
$$\int \frac{xe^x}{(1+x)^2} dx = \int \frac{e^x(x+1-1)}{(1+x)^2} dx$$

$$= \int e^x \left(\frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx$$

यदि $[f(x)] = \frac{1}{x+1}$ तो $\int e^x f(x) + f(x) dx$ यदि $f(x) = \frac{1}{(x+1)^2}$]

$$= \int \frac{1}{x+1} \cdot e^x dx - \int \frac{1}{(x+1)^2} e^x dx$$

$$= \frac{1}{x+1} e^x - \int -\frac{1}{(x+1)^2} e^x dx - \int \frac{1}{(x+1)^2} e^x dx$$

$$= \frac{e^x}{x+1} + \int \frac{e^x}{(x+1)^2} dx - \int \frac{e^x}{(x+1)^2} dx$$

$$= \frac{e^x}{x+1} + C.$$

उत्तर

प्रश्न 18. $e^x \left(\frac{1 + \sin x}{1 + \cos x} \right).$

हल :
$$\int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx = \int \frac{e^x \left(1 + 2 \sin \frac{x}{2} \cos \frac{x}{2} \right)}{2 \cos^2 \frac{x}{2}} dx$$

$$= \int e^x \left(\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx$$

$$= \int e^x \tan \frac{x}{2} dx + \frac{1}{2} \int e^x \sec^2 \frac{x}{2} dx \quad \dots(i)$$

खण्डशः समाकलन करने पर,

$$\therefore \int \tan \frac{x}{2} e^x dx = \left(\tan \frac{x}{2} \right) \cdot e^x - \int \frac{1}{2} \sec^2 \frac{x}{2} e^x dx$$

$$= e^x \tan \frac{x}{2} - \frac{1}{2} \int \sec^2 \frac{x}{2} e^x dx$$

यह मान समीकरण (i) में रखने पर

$$= e^x \tan \frac{x}{2} - \frac{1}{2} \int \sec^2 \frac{x}{2} e^x dx + \frac{1}{2} \int e^x \cdot \sec^2 \frac{x}{2} dx$$

$$= e^x \tan \frac{x}{2} + C.$$

उत्तर

प्रश्न 19. $e^x \left(\frac{1}{x} - \frac{1}{x^2} \right).$

हल :
$$\int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \int e^x \cdot \frac{1}{x} dx - \int e^x \cdot \frac{1}{x^2} dx$$

अब $e^x \cdot \frac{1}{x}$ का खण्डशः समाकलन करने पर,

$$\begin{aligned} &= \frac{1}{x} \cdot e^x - \int \frac{-1}{x^2} \cdot e^x dx - \int e^x \cdot \frac{1}{x^2} dx \\ &= \frac{1}{x} \cdot e^x + \int \frac{1}{x^2} \cdot e^x dx - \int e^x \cdot \frac{1}{x^2} dx \\ &= \frac{e^x}{x} + C \end{aligned}$$

उत्तर

प्रश्न 20. $\frac{(x-3)e^x}{(x-1)^3}$.

हल : माना कि

$$\begin{aligned} I &= \int \frac{(x-3)e^x}{(x-1)^3} dx = \int e^x \left[\frac{x-1-2}{(x-1)^3} \right] dx \\ &= \int e^x \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right] dx \\ &= \int \frac{1}{(x-1)^2} \cdot e^x dx - \int \frac{2}{(x-1)^3} \cdot e^x dx \end{aligned}$$

यहाँ $e^x \cdot \frac{1}{(x-1)^2} = t$ रखने पर

$$\left\{ e^x \cdot \frac{1}{(x-1)^2} + e^x \cdot \left(\frac{-2}{(x-1)^3} \right) \right\} dx = dt$$

या $e^x \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right] dx = dt$

∴

$$I = \int dt = t + C = \frac{e^x}{(x-1)^2} + C.$$

उत्तर

प्रश्न 21. $e^{2x} \sin x$.

हल : मान लीजिए

खण्डशः समाकलन करने पर,

$$I = \int e^{2x} \sin x dx$$

$$\begin{aligned} I &= e^{2x}(-\cos x) - \int 2e^{2x} \cdot (-\cos x) dx \\ &= -e^{2x} \cos x + 2 \int e^{2x} \cos x dx \end{aligned}$$

पुनः खण्डशः समाकलन करने पर

$$\begin{aligned} I &= -e^{2x} \cos x + 2[e^{2x} \sin x - \int 2e^{2x} \sin x dx] \\ &= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x dx \\ I &= -e^{2x} \cos x + 2e^{2x} \sin x - 4I \end{aligned}$$

या

$$5I = e^{2x} \cdot 2 \sin x - e^{2x} \cos x$$

∴

$$I = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C.$$

उत्तर

प्रश्न 22. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

हल : $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

मान लीजिए $x = \tan \theta$ हो, तब $dx = \sec^2 \theta d\theta$

$$\begin{aligned} &= \int \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right) \sec^2 \theta d\theta \\ &= \int \sin^{-1}(\sin 2\theta) \sec^2 \theta d\theta \\ &= \int 2\theta \sec^2 \theta d\theta = 2 \int \theta \sec^2 \theta d\theta \end{aligned}$$

खण्डशः समाकलन करने पर

$$\begin{aligned} &= 2[\theta \tan \theta - \int 1 \cdot \tan \theta d\theta] \\ &= 2[\theta \tan \theta + \log |\cos \theta|] + C \end{aligned}$$

यहाँ $\tan \theta = x$, $\cos \theta = \frac{1}{\sqrt{1+x^2}}$ को रखने पर,

$$\begin{aligned} &= 2\left[x \tan^{-1} x + \log \left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C \\ &= 2x \tan^{-1} x - 2 \cdot \frac{1}{2} \log(1+x^2) + C \\ &= 2x \tan^{-1} x - \log(1+x^2) + C. \end{aligned}$$

उत्तर

प्रश्न 23 व 24 में सही उत्तर का चयन कीजिए—

प्रश्न 23. $\int x^2 e^{x^3} dx$ बराबर है—

(A) $\frac{1}{3} e^{x^3} + C$

(B) $\frac{1}{3} e^{x^2} + C$

(C) $\frac{1}{2} e^{x^3} + C$

(D) $\frac{1}{2} e^{x^2} + C$

हल : $\int x^2 e^{x^3} dx$

$x^3 = t$ रखने पर,

$\therefore$

$$\begin{aligned} 3x^2 dx &= dt \\ &= \frac{1}{3} \int e^t dt = \frac{1}{3} e^t + C \\ &= \frac{1}{3} e^{x^3} \end{aligned}$$

अतः विकल्प (A) सही है।

उत्तर

प्रश्न 24. $\int e^x \sec x(1 + \tan x) dx$ बराबर है—

(A) $e^x \cos x + C$

(B) $e^x \sec x + C$

(C) $e^x \sin x + C$

(D) $e^x \tan x + C$

हल : $\int e^x \sec x(1 + \tan x) dx = \int e^x \sec x dx + \int e^x \sec x \tan x dx$

अब प्रथम समाकलन में खण्डशः समाकलन का प्रयोग करने पर,

$$= \sec x \int e^x - \int \sec x \tan x \int e^x dx + \int e^x \sec x \tan x dx$$

$$= \sec x \cdot e^x - \int e^x \sec x \tan x dx + \int e^x \cdot \sec x \tan x dx$$

$$= e^x \sec x + C.$$

अतः विकल्प (B) सही है।

उत्तर

प्रश्नावली 7.7

प्रश्न 1 से 9 तक के प्रश्नों के फलनों का समाकलन कीजिए—

प्रश्न 1. $\sqrt{4-x^2}$.

हल :

$$\int \sqrt{4-x^2} dx$$

$$= \int \sqrt{(2)^2 - x^2} dx$$

$$= \frac{x\sqrt{4-x^2}}{2} + \frac{4}{2} \sin^{-1} \frac{x}{2} + C$$

$$\left[\because \int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \text{ से } \right]$$

$$= \frac{x\sqrt{4-x^2}}{2} + 2 \sin^{-1} \frac{x}{2} + C$$

$$= \frac{1}{2} x\sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} + C.$$

उत्तर

प्रश्न 2. $\sqrt{1-4x^2}$.

हल :

$$\int \sqrt{1-4x^2} dx = 2 \int \sqrt{\frac{1}{4} - x^2} dx$$

$\therefore$

$$= 2 \left[\frac{x}{2} \sqrt{\frac{1}{4} - x^2} + \frac{1}{4 \cdot 2} \sin^{-1} \left(\frac{x}{\frac{1}{2}} \right) \right] + C$$

$$\left[\because \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$$

$$= x \frac{\sqrt{1-4x^2}}{2} + \frac{1}{4} \sin^{-1} 2x + C$$

$$= \frac{x}{2} \sqrt{1-4x^2} + \frac{1}{4} \sin^{-1} 2x + C$$

$$= \frac{1}{4} \sin^{-1} 2x + \frac{1}{2} x\sqrt{1-4x^2} + C.$$

उत्तर

प्रश्न 3. $\sqrt{x^2 + 4x + 6}$.

हल :
$$\begin{aligned} \int \sqrt{x^2 + 4x + 6} \, dx &= \int \sqrt{x^2 + 4x + 4 + 2} \, dx = \int \sqrt{(x+2)^2 + 2} \, dx \\ &= \frac{(x+2)\sqrt{(x+2)^2 + 2}}{2} \\ &\quad + \frac{2}{2} \log |(x+2) + \sqrt{(x+2)^2 + 2}| + C \\ \left[\because \int \sqrt{x^2 + a^2} \, dx &= \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| \text{ से } \right] \\ &= \frac{1}{2}(x+2)\sqrt{x^2 + 4x + 6} \\ &\quad + \log |(x+2) + \sqrt{x^2 + 4x + 6}| + C. \text{ उत्तर} \end{aligned}$$

प्रश्न 4. $\sqrt{x^2 + 4x + 1}$.

हल :
$$\begin{aligned} \int \sqrt{x^2 + 4x + 1} \, dx &= \int \sqrt{x^2 + 4x + 4 - 3} \, dx \\ &= \int \sqrt{(x+2)^2 - 3} \, dx \\ &= \frac{(x+2)\sqrt{(x+2)^2 - 3}}{2} \\ &\quad - \frac{3}{2} \log |(x+2) + \sqrt{(x+2)^2 - 3}| + C \\ \left[\because \int \sqrt{x^2 - a^2} \, dx &= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| \right] \\ &= \frac{1}{2}(x+2)\sqrt{x^2 + 4x + 1} \\ &\quad - \frac{3}{2} \log |(x+2)\sqrt{x^2 + 4x + 1}| + C. \text{ उत्तर} \end{aligned}$$

प्रश्न 5. $\sqrt{1 - 4x - x^2}$.

हल :
$$\begin{aligned} \int \sqrt{1 - 4x - x^2} \, dx &= \int \sqrt{1 - (x^2 + 4x + 4) + 4} \, dx \\ &= \int \sqrt{5 - (x+2)^2} \, dx \\ &= \frac{1}{2}(x+2)\sqrt{5 - (x+2)^2} + \frac{5}{2} \sin^{-1} \frac{x+2}{\sqrt{5}} + C \\ \left[\because \int \sqrt{a^2 - x^2} \, dx &= \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right] \\ &= \frac{1}{2}(x+2)\sqrt{1 - 4x - x^2} + \frac{5}{2} \sin^{-1} \frac{x+2}{\sqrt{5}} + C \\ &= \frac{5}{2} \sin^{-1} \left(\frac{x+2}{\sqrt{5}} \right) + \frac{x+2}{2} \sqrt{1 - 4x - x^2} + C. \text{ उत्तर} \end{aligned}$$

प्रश्न 6. $\sqrt{x^2 + 4x - 5}$.

हल :
$$\begin{aligned}\int \sqrt{x^2 + 4x - 5} \, dx &= \int \sqrt{x^2 + 4x + 4 - 9} \, dx \\ &= \int \sqrt{(x+2)^2 - 9} \, dx \\ &= \int \sqrt{(x+2)^2 - (3)^2} \, dx \\ &\left[\because \int \sqrt{x^2 - a^2} \, dx = \int \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| \right] \\ &= \frac{1}{2}(x+2)\sqrt{(x+2)^2 - 9} \\ &\quad - \frac{9}{2} \log|(x+2) + \sqrt{(x+2)^2 - 9}| + C \\ &= \frac{1}{2}(x+2)\sqrt{x^2 + 4x - 5} \\ &\quad - \frac{9}{2} \log|(x+2) + \sqrt{x^2 + 4x - 5}| + C. \text{ उत्तर}\end{aligned}$$

प्रश्न 7. $\sqrt{1 + 3x - x^2}$.

हल :
$$\begin{aligned}\int \sqrt{1 + 3x - x^2} \, dx &= \int \sqrt{1 - \left(x^2 - 3x + \frac{9}{4}\right) + \frac{9}{4}} \, dx \\ &= \int \sqrt{\frac{13}{4} - \left(x - \frac{3}{2}\right)^2} \, dx \\ &= \frac{1}{2}\left(x - \frac{3}{2}\right) \sqrt{\frac{13}{4} - \left(x - \frac{3}{2}\right)^2} + \frac{13}{8} \sin^{-1} \frac{x - \frac{3}{2}}{\sqrt{\frac{13}{4}}} + C \\ &\left[\because \int \sqrt{a^2 - x^2} \, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right] \\ &= \frac{2x-3}{4} \sqrt{1 + 3x - x^2} + \frac{13}{8} \sin^{-1} \left(\frac{2x-3}{\sqrt{13}} \right) + C. \text{ उत्तर}\end{aligned}$$

प्रश्न 8. $\sqrt{x^2 + 3x}$.

हल :
$$\begin{aligned}\int \sqrt{x^2 + 3x} \, dx &= \int \sqrt{x^2 + 3x + \frac{9}{4} - \frac{9}{4}} \, dx \\ &= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \frac{9}{4}} \, dx\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(x + \frac{3}{2} \right) \sqrt{\left(x + \frac{3}{2} \right)^2 - \frac{9}{4}} \\
&\quad - \frac{9}{8} \log \left| \left(x + \frac{3}{2} \right) + \sqrt{\left(x + \frac{3}{2} \right)^2 - \frac{9}{4}} \right| + C \\
\left[\because \int \sqrt{x^2 - a^2} dx = \int \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| \right] \\
&= \frac{2x+3}{4} \sqrt{x^2 + 3x} - \frac{9}{8} \log \left| x + \frac{3}{2} + \sqrt{x^2 + 3x} \right| + C. \quad \text{उत्तर}
\end{aligned}$$

प्रश्न 9. $\sqrt{1 + \frac{x^2}{9}}$.

हल :

$$\begin{aligned}
\int \sqrt{1 + \frac{x^2}{9}} dx &= \int \frac{\sqrt{x^2 + 9}}{3} dx \\
&= \frac{1}{3} \int \sqrt{x^2 + 9} dx \\
\because \int \sqrt{x^2 + a^2} dx &= \int \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| \\
&= \frac{1}{3} \left[\frac{x\sqrt{x^2 + 9}}{2} + \frac{3}{2} \log |x + \sqrt{x^2 + 9}| \right] + C \\
&= \frac{1}{6} x\sqrt{x^2 + 9} + \frac{3}{2} \log |x + \sqrt{x^2 + 9}| + C. \quad \text{उत्तर}
\end{aligned}$$

प्रश्न 10 व 11 में सही उत्तर का चयन कीजिए—

प्रश्न 10. $\int \sqrt{1 + x^2} dx$ बराबर है—

- (A) $\frac{x}{2} \sqrt{1 + x^2} + \frac{1}{2} \log |x + \sqrt{1 + x^2}| + C$ (B) $\frac{2}{3} (1 + x^2)^{3/2} + C$
 (C) $\frac{2}{3} x(1 + x^2)^{3/2} + C$
 (D) $\frac{x^2}{2} \sqrt{1 + x^2} + \frac{x^2}{2} \log |x + \sqrt{1 + x^2}| + C$

हल :

$$\int \sqrt{1 + x^2} dx = \int \sqrt{(1)^2 + x^2} dx$$

सूत्रानुसार,

$$\begin{aligned}
\int \sqrt{a^2 + x^2} dx &= \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log |x + \sqrt{a^2 + x^2}| + C \\
&= \frac{x}{2} \sqrt{1 + x^2} + \frac{1}{2} \log |x + \sqrt{1 + x^2}| + C
\end{aligned}$$

अतः विकल्प (A) सही है।

उत्तर

प्रश्न 11. $\int \sqrt{x^2 - 8x + 7} dx$ बराबर है—

- (A) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} + 9 \log |x-4 + \sqrt{x^2 - 8x + 7}| + C$
 (B) $\frac{1}{2}(x+4)\sqrt{x^2 - 8x + 7} + 9 \log |x+4 + \sqrt{x^2 - 8x + 7}| + C$
 (C) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} - 3\sqrt{2} \log |x-4 + \sqrt{x^2 - 8x + 7}| + C$
 (D) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} - \frac{9}{2} \log |x-4 + \sqrt{x^2 - 8x + 7}| + C$

हल :
$$\int \sqrt{x^2 - 8x + 7} dx = \int \sqrt{x^2 - 8x + 16 + 7 - 16} dx$$

$$= \int \sqrt{(x-4)^2 - 9} dx$$

अब सूत्र $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$ के प्रयोग से,

$$= \frac{x-4}{2} \sqrt{x^2 - 8x + 7} - \frac{9}{2} \log |(x-4) + \sqrt{x^2 - 8x + 7}| + C$$

अतः विकल्प (D) सही है।

उत्तर

प्रश्नावली 7.8

योगों की सीमा के रूप में निम्नलिखित निश्चित समाकलनों का मान ज्ञात कीजिए—

प्रश्न 1. $\int_a^b x dx$

हल : यहाँ

$$f(x) = x \text{ और } b - a = nh$$

$$f(x) = x, f(a+h) = a+h, f(a+2h) = a+2h, \dots,$$

$$f(a+(n-1)h) = a+(n-1)h$$

$$\begin{aligned} \therefore \int_a^b x dx &= \lim_{h \rightarrow 0} h[a + (a+h) + (a+2h) + (a+3h) + \dots + (a+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h[na + h(1+2+3+\dots+(n-1))] \\ &= \lim_{h \rightarrow 0} h \left[na + h \times \frac{n(n-1)}{2} \right] \\ &= \lim_{h \rightarrow 0} \left[(nh)a + \frac{(nh)(nh-h)}{2} \right] \\ &= \frac{(b-a)a + (b-a)(b-a-0)}{2} = \frac{2ab - 2a^2 + b^2 + a^2 - 2ab}{2} \\ &= \frac{b^2 - a^2}{2} \end{aligned}$$

उत्तर

प्रश्न 2. $\int_0^5 (x+1) dx$.

हल : $\int_0^5 (x+1) dx$

दिया है :

$$\begin{aligned} f(x) &= x+1, a=0, b=5 \\ f(a) &= a+1, f(a+h) = a+h+1 \\ f(a+2h) &= a+2h+1 \end{aligned}$$

$$\begin{aligned} f(a+\overline{n-1}h) &= a+\overline{n-1}h+1 \\ nh &= b-a = 5-0 = 5 \end{aligned}$$

$$\begin{aligned} \therefore \int_0^5 (x+1) dx &= \lim_{h \rightarrow 0} h(a+1) + (a+h+1) + (a+2h+1) + \dots \\ &\quad + (a+\overline{n-1}h+1) \\ &= \lim_{h \rightarrow 0} h[n(a+1) + h(1+2+3+\dots+\overline{n-1})] \\ &= \lim_{h \rightarrow 0} h \left[n(a+1) + h \cdot \frac{(n-1)n}{2} \right] \\ &= \lim_{h \rightarrow 0} \left[nh(a+1) + \frac{(nh-h)nh}{2} \right] \quad \left[\begin{array}{l} a=0 \text{ तथा} \\ nh=5 \end{array} \right] \\ &= 5 \times 1 + \frac{5 \times 5}{2} = 5 + \frac{25}{2} = \frac{35}{2} \end{aligned}$$

उत्तर

प्रश्न 3. $\int_2^3 x^2 dx$.

हल :

$$\int_2^3 x^2 dx$$

यहाँ $nh = 3 - 2 = 1$ और $f(x) = x^2, a = 2, b = 3$

$$f(x) = x^2, f(2) = 2^2, f(2+h) = (2+h)^2$$

$$f(2+2h) = (2+2h)^2, \dots, f(2+\overline{n-1}h) = [2+(n-1)h]^2$$

$$\begin{aligned} \therefore \int_a^b f(x) dx &= \lim_{h \rightarrow 0} h[f(a) + f(a+h) + f(a+2h) + \dots \\ &\quad + f(a+\overline{n-1}h)] \end{aligned}$$

$$\begin{aligned} \int_2^3 x^2 dx &= \lim_{h \rightarrow 0} h[2^2 + (2+h)^2 + (2+2h)^2 + (2+3h)^2 + \dots \\ &\quad + (2+\overline{n-1}h)^2] \end{aligned}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} h\{2^2 + (2^2 + 2.2h + h^2) \\ &\quad + [2^2 + 2(2.2h) + (2h)^2] + [2^2 + 2(2.3h) + (3h)^2] + \dots \\ &\quad + [2^2 + 2.2(n-1)h + (\overline{n-1}h)^2]\} \end{aligned}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} h\{[n.2^2 + 4h(1+2+3+\dots+\overline{n-1}) \\ &\quad + h^2[1+2^2+3^2+\dots+(n-1)^2]]\} \end{aligned}$$

$$= \lim_{h \rightarrow 0} \left[nh.2^2 + 4h \cdot \frac{(n-1)n}{2} + h^2 \frac{(n-1)n(2n-1)}{6} \right]$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \left[nh.4 + \frac{4}{2} \cdot (nh-h)nh + \frac{(nh-h)(nh)(2nh-h)}{6} \right] \\
&= 1.4 + 2.1.1 + \frac{1.1.2}{6} = 6 + \frac{1}{3} = \frac{19}{3}.
\end{aligned}$$

उत्तर

प्रश्न 4. $\int_1^4 (x^2 - x) dx$.

हल : $\int_1^4 (x^2 - x) dx$

यहाँ $a = 1, b = 4, f(x) = x^2 - x, nh = b - a = 4 - 1 = 3$

$$\begin{aligned}
\therefore \int_1^4 (x^2 - x) dx &= \lim_{h \rightarrow 0} h[f(1) + f(1+h) + f(1+2h) + \dots \\
&\quad + f\{(1+(n-1)h)\}] \\
&= \lim_{h \rightarrow 0} h[(1-1) + \{(1+h)^2 - (1+h)\} \\
&\quad + \{(1+2h)^2 - (1+2h)\} + \dots \\
&\quad + \{(1+(n-1)h)^2 - \{1+(n-1)h\}\}] \\
&= \lim_{h \rightarrow 0} h[(1+h^2+2h-1-h) \\
&\quad + (1+4h^2+4h-1-2h) + \dots \\
&\quad + \{1+(n-1)^2h^2+2(n-1)h-1-(n-1)h\}] \\
&= \lim_{h \rightarrow 0} h[(h^2+h) + (4h^2+2h) + \dots \\
&\quad + \{(n-1)h^2+(n-1)h\}] \\
&= \lim_{h \rightarrow 0} h[h^2(1^2+2^2+3^2+\dots+(n-1)^2 \\
&\quad + h\{1+2+3+\dots+(n-1)\})] \\
&= \lim_{h \rightarrow 0} \left[h^3 \cdot \frac{n(n-1)(2n-1)}{6} + h^2 \frac{n(n-1)}{2} \right] \\
&= \lim_{h \rightarrow 0} \left[\frac{nh(nh-h)(2nh-h)}{6} + \frac{nh(nh-h)}{2} \right] \\
&= \frac{3 \cdot (3-0)(6-0)}{6} + \frac{3(3-0)}{2} \\
&= 9 + \frac{9}{2} = \frac{27}{2}.
\end{aligned}$$

उत्तर

प्रश्न 5. $\int_{-1}^1 e^x dx$.

हल : $\int_{-1}^1 e^x dx$

जबकि $\int_a^b f(x) dx = \lim_{h \rightarrow 0} h[f(a) + f(a+2h) + f(a+3h) + \dots + f(a+(n-1)h)]$

दिया है : $a = -1, b = 1, b - a = 2 = nh$

$$\begin{aligned}
\therefore f(a) &= e^x \\
f(-1) &= e^{-1}, f(-1+h) = e^{-1+h} \\
f(-1+2h) &= e^{-1+2h} \\
f(-1+(n-1)h) &= e^{-1+(n-1)h}
\end{aligned}$$

$$\begin{aligned}
\therefore \int_{-1}^1 e^x dx &= \lim_{h \rightarrow 0} h[e^{-1} + e^{-1+h} + e^{-1+2h} + \dots + e^{-1+(n-1)h}] \\
&= \lim_{h \rightarrow 0} h e^{-1} [1 + e^h + e^{2h} + \dots + e^{(n-1)h}] \\
&= \lim_{h \rightarrow 0} h e^{-1} \left[\frac{1 - e^{nh}}{1 - e^h} \right] \\
&\quad \left[\because a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \text{ से } \right] \\
&= \lim_{h \rightarrow 0} e^{-1} \frac{(1 - e^{nh})}{\left(\frac{e^h - 1}{h} \right)} = \frac{-e^{-1}(1 - e^2)}{1} \\
&\quad \left[\because nh = 2, \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1 \right] \\
&= \frac{-1 + e^2}{e} = e - \frac{1}{e}.
\end{aligned}$$

उत्तर

प्रश्न 6. $\int_0^4 (x + e^{2x}) dx$.

हल : हमें ज्ञात है :

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h[f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]$$

यहाँ $nh = b - a$

दिया है : $a = 0, b = 4, nh = 4 - 0 = 4$

$$\begin{aligned}
\therefore \int_0^4 (x + e^{2x}) dx &= \lim_{h \rightarrow 0} h[f(0) + f(0+h) + f(0+2h) + \dots \\
&\quad + f(0+(n-1)h)] \\
&= \lim_{h \rightarrow 0} h[(0 + e^0) + (h + e^{2h}) + (2h + e^{2(2h)}) + \dots \\
&\quad + (n-1)h + e^{2(n-1)h}] \\
&= \lim_{h \rightarrow 0} h[(1 + 2 + 3 + \dots + (n-1)) \\
&\quad + (1 + e^{2h} + e^{2(2h)} + \dots + e^{2(n-1)h})] \\
&= \lim_{h \rightarrow 0} \left[h^2 \cdot \frac{n(n-1)}{2} + h \cdot \frac{1(1 - e^{2nh})}{1 - e^{2h}} \right] \\
&= \lim_{h \rightarrow 0} \left[\frac{nh(nh-h)}{2} + \frac{1}{\frac{1 - e^{2h}}{2h}} (1 - e^{2nh}) \right] \\
&\quad \left[\because \lim_{h \rightarrow 0} \frac{e^{2h} - 1}{2h} = 1 \right] \\
&= \frac{4(4-0)}{2} - \frac{1}{2}(1 - e^8) = 8 - \frac{1}{2}(1 - e^8) \\
&= \frac{15}{2} + \frac{e^8}{2} = \frac{1}{2}(15 + e^8).
\end{aligned}$$

उत्तर

प्रश्नावली 7.9

प्रश्न 1 से 20 तक के प्रश्नों में निश्चित समाकलनों का मान ज्ञात कीजिए—

प्रश्न 1. $\int_{-1}^1 (x+1) dx$.

हल : ज्ञात है :

$$\begin{aligned}\int_{-1}^1 (x+1) dx &= \left[\frac{x^2}{2} + x \right]_{-1}^1 = \left(\frac{1}{2} + 1 \right) - \left(\frac{1}{2} - 1 \right) \\ &= \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2.\end{aligned}$$

उत्तर

प्रश्न 2. $\int_2^3 \frac{1}{x} dx$.

हल : ज्ञात है :

$$\int_2^3 \frac{1}{x} dx = [\log x]_2^3 = \log 3 - \log 2 = \log \frac{3}{2}.$$

उत्तर

प्रश्न 3. $\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$.

हल : ज्ञात है : $\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$

$$\begin{aligned}&= \left[4 \cdot \frac{x^4}{4} - 5 \cdot \frac{x^3}{3} + 6 \cdot \frac{x^2}{2} + 9x \right]_1^2 \\ &= \left[x^4 - \frac{5}{3}x^3 + 3x^2 + 9x \right]_1^2 \\ &= \left[\left(16 - \frac{5}{3} \times 8 + 3 \times 4 + 9 \times 2 \right) - \left(1 - \frac{5}{3} + 3 + 9 \right) \right] \\ &= \left[16 - \frac{40}{3} + 12 + 18 \right] - \left[13 - \frac{5}{3} \right] \\ &= \left[46 - \frac{40}{3} \right] - \left[13 - \frac{5}{3} \right] = 33 - \frac{40}{3} + \frac{5}{3} \\ &= 33 - \frac{35}{3} = \frac{99 - 35}{3} = \frac{64}{3}.\end{aligned}$$

उत्तर

प्रश्न 4. $\int_0^{\pi/4} \sin 2x dx$.

हल : ज्ञात है :

$$\begin{aligned}\int_0^{\pi/4} \sin 2x dx &= -\frac{1}{2} [\cos 2x]_0^{\pi/4} \\ &= -\frac{1}{2} [0 - 1] = \frac{1}{2}.\end{aligned}$$

उत्तर

प्रश्न 5. $\int_0^{\pi/2} \cos 2x dx$.

हल : ज्ञात है :

$$\begin{aligned}\int_0^{\pi/2} \cos 2x \, dx &= \frac{1}{2} [\sin 2x]_0^{\pi/2} = \frac{1}{2} [\sin 2 \times \frac{\pi}{2} - \sin 0] \\ &= \frac{1}{2} [\sin \pi - 0] = \frac{1}{2} (0 - 0) = 0.\end{aligned}$$

उत्तर

प्रश्न 6. $\int_4^5 e^x \, dx$.

हल : ज्ञात है :

$$\int_4^5 e^x \, dx = [e^x]_4^5 = e^5 - e^4 = e^4(e - 1).$$

उत्तर

प्रश्न 7. $\int_0^{\pi/4} \tan x \, dx$.

हल : ज्ञात है :

$$\begin{aligned}\int_0^{\pi/4} \tan x \, dx &= [\log \cos x]_0^{\pi/4} \\ &= -\left[\log \cos \frac{\pi}{4} - \log \cos 0\right] \\ &= -\left[\log \frac{1}{\sqrt{2}} - 0\right] = \log \sqrt{2} = \log (2)^{1/2} = \frac{1}{2} \log 2.\end{aligned}$$

उत्तर

प्रश्न 8. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x \, dx$.

हल : ज्ञात है :

$$\begin{aligned}\int_{\pi/6}^{\pi/4} \operatorname{cosec} x \, dx &= [\log (\operatorname{cosec} x - \cot x)]_{\pi/6}^{\pi/4} \\ &= \log \left(\operatorname{cosec} \frac{\pi}{4} - \cot \frac{\pi}{4} \right) - \log \left(\operatorname{cosec} \frac{\pi}{6} - \cot \frac{\pi}{6} \right) \\ &= \log (\sqrt{2} - 1) - \log (2 - \sqrt{3}) \\ &= \log \left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}} \right).\end{aligned}$$

उत्तर

प्रश्न 9. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

हल : ज्ञात है :

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} &= [\sin^{-1} x]_0^1 = \sin^{-1} 1 - \sin^{-1} 0 \\ &= \sin^{-1} \left(\sin \frac{\pi}{2} \right) - \sin^{-1} (\sin 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.\end{aligned}$$

उत्तर

प्रश्न 10. $\int_0^1 \frac{dx}{1+x^2}$.

हल : हमें ज्ञात है :

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &= [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 \\ &= \tan^{-1} \left(\tan \frac{\pi}{4} \right) - \tan^{-1} (\tan 0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.\end{aligned}$$

उत्तर

प्रश्न 11. $\int_2^3 \frac{dx}{x^2 - 1}$.

हल : ज्ञात है :

$$\begin{aligned}\int_2^3 \frac{dx}{x^2 - 1} &= \frac{1}{2} \left[\log \frac{x-1}{x+1} \right]_2^3 = \frac{1}{2} \left[\log \frac{2}{4} - \log \frac{1}{3} \right] \\ &= \frac{1}{2} \left[\log \left(\frac{1}{2} \times \frac{3}{1} \right) \right] = \frac{1}{2} \log \frac{3}{2}.\end{aligned}$$

उत्तर

प्रश्न 12. $\int_0^{\pi/2} \cos^2 x \, dx$.

हल : मान लीजिए

$$I = \int_0^{\pi/2} \cos^2 x \, dx \quad \dots(i)$$

$$= \int_0^{\pi/2} \cos^2 \left(\frac{\pi}{2} - x \right) dx \quad \left[\because \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right]$$

$$I = \int_0^{\pi/2} \sin^2 x \, dx \quad \dots(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$\begin{aligned}2I &= \int_0^{\pi/2} \cos^2 x \, dx + \int_0^{\pi/2} \sin^2 x \, dx \\ &= \int_0^{\pi/2} (\cos^2 x + \sin^2 x) \, dx \\ &= \int_0^{\pi/2} 1 \, dx = [x]_0^{\pi/2}\end{aligned}$$

$$2I = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}.$$

उत्तर

प्रश्न 13. $\int_2^3 \frac{x \, dx}{x^2 + 1}$.

हल : ज्ञात है : $\int_2^3 \frac{x \, dx}{x^2 + 1}$ मान लीजिए $x^2 + 1 = t$ हो, तब $\therefore 2x \, dx = dt$

$$\int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log t = \frac{1}{2} \log (x^2 + 1)$$

$\therefore$

$$\begin{aligned}\int_2^3 \frac{x \, dx}{x^2 + 1} &= \frac{1}{2} \left[\log (x^2 + 1) \right]_2^3 \\ &= \frac{1}{2} [\log (3^2 + 1) - \log (2^2 + 1)] \\ &= \frac{1}{2} [\log 10 - \log 5] \\ &= \frac{1}{2} \log \frac{10}{5} = \frac{1}{2} \log 2.\end{aligned}$$

उत्तर

प्रश्न 14. $\int_0^1 \frac{2x+3}{5x^2+1} dx$.

हल : ज्ञात है : $\int_0^1 \frac{2x+3}{5x^2+1} dx$

$$\begin{aligned} \int \frac{2x+3}{5x^2+1} dx &= \frac{1}{5} \int \frac{10x}{5x^2+1} dx + 3 \int \frac{1}{5x^2+1} dx \\ &= \frac{1}{5} \int \frac{10x}{5x^2+1} dx + \frac{3}{5} \int \frac{1}{x^2 + \frac{1}{5}} dx \end{aligned}$$

प्रथम समाकलन में माना $5x^2 + 1 = t$ तब $\therefore 10x dx = dt$

$$= \frac{1}{5} \int \frac{dt}{t} + \frac{3}{5} \int \frac{1}{(x^2) + \left(\frac{1}{\sqrt{5}}\right)^2} dx$$

$$= \frac{1}{5} \log t + \frac{3}{5} \times \sqrt{5} \tan^{-1} \left(\frac{x}{1/\sqrt{5}} \right)$$

$$= \frac{1}{5} \log (5x^2 + 1) + \frac{3}{\sqrt{5}} \tan^{-1} (x\sqrt{5})$$

$$\therefore \int_0^1 \frac{2x+3}{5x^2+1} dx = \left[\frac{1}{5} \log (5x^2 + 1) + \frac{3}{\sqrt{5}} \tan^{-1} (\sqrt{5}x) \right]_0^1$$

$$= \left[\frac{1}{5} \log (5 + 1) + \frac{3}{\sqrt{5}} \tan^{-1} (\sqrt{5}) \right]$$

$$- \left[\frac{1}{5} \log (0 + 1) + \frac{3}{\sqrt{5}} \tan^{-1} 0 \right]$$

$$= \frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5}. \quad (\because \log 1 = 0) \text{ उत्तर}$$

प्रश्न 15. $\int_0^1 x e^{x^2} dx$.

हल : $\int x e^{x^2} dx$, मान लीजिए $x^2 = t$ तब $2x dx = dt$

$$= \frac{1}{2} \int e^t dt = \frac{1}{2} e^t = \frac{1}{2} e^{x^2}$$

$$\therefore \int_0^1 x e^{x^2} dx = \left[\frac{1}{2} e^{x^2} \right]_0^1 = \frac{1}{2} e^1 - \frac{1}{2} e^0 = \frac{1}{2} (e - 1).$$

उत्तर

प्रश्न 16. $\int_1^2 \frac{5x^2}{x^2 + 4x + 3} dx$.

हल : $\int \frac{5x^2}{x^2 + 4x + 3} dx$

$5x^2$ को $x^2 + 4x + 3$ से भाग देने पर,

$$\therefore \frac{5x^2}{x^2 + 4x + 3} = 5 - \frac{20x + 15}{x^2 + 4x + 3}$$

$$\text{अब } \frac{20x + 15}{x^2 + 4x + 3} = \frac{20x + 15}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

$$\therefore 20x + 15 = A(x+1) + B(x+3)$$

$x = -3$ लेने पर,

$$-60 + 15 = A(-2)$$

$$A = \frac{45}{2}$$

तथा $x = -1$ लेने पर,

$$-20 + 15 = B \times 2$$

$$\therefore B = -\frac{5}{2}$$

$$\therefore \frac{20x + 15}{x^2 + 4x + 3} = \frac{45}{2(x+3)} - \frac{5}{2(x+1)}$$

$$\begin{aligned} \text{अब } \int \frac{5x^2}{x^2 + 4x + 3} dx &= \int \left[5 - \frac{45}{2(x+3)} + \frac{5}{2(x+1)} \right] dx \\ &= 5x - \frac{45}{2} \log |x+3| + \frac{5}{2} \log |x+1| \end{aligned}$$

$$\begin{aligned} \therefore \int_1^2 \frac{5x^2}{x^2 + 4x + 3} dx &= \left[5x - \frac{45}{2} \log |x+3| + \frac{5}{2} \log |x+1| \right]_1^2 \\ &= \left[\left(10 - \frac{45}{2} \log 5 + \frac{5}{2} \log 3 \right) - \left(5 - \frac{45}{2} \log 4 + \frac{5}{2} \log 2 \right) \right] \\ &= 5 - \frac{45}{2} \log \frac{5}{4} + \frac{5}{2} \log \frac{3}{2} \\ &= 5 - \frac{5}{2} \left(9 \log \frac{5}{4} - \log \frac{3}{2} \right). \end{aligned}$$

उत्तर

प्रश्न 17. $\int_0^{\pi/4} (2 \sec^2 x + x^3 + 2) dx.$

हल : ज्ञात है :

$$\int_0^{\pi/4} (2 \sec^2 x + x^3 + 2) dx = 2 \int_0^{\pi/4} \sec^2 x dx + \int_0^{\pi/4} x^3 dx + 2 \int_0^{\pi/4} dx$$

$$= \left[2 \tan x + \frac{x^4}{4} + 2x \right]_0^{\pi/4}$$

$$= \left(2 \tan \frac{\pi}{4} + \frac{\pi^4}{1024} + \frac{\pi}{2} \right) - 0$$

$$= 2 + \frac{\pi^4}{1024} + \frac{\pi}{2}$$

$$= \frac{\pi^4}{1024} + \frac{\pi}{2} + 2.$$

उत्तर

प्रश्न 18. $\int_0^\pi \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx.$

हल : ज्ञात है :

$$\begin{aligned} \int_0^\pi \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx &= - \int_0^\pi \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx \\ &= - \int_0^\pi \cos x \, dx = - [\sin x]_0^\pi \\ &= - [\sin \pi + \sin 0] = 0. \end{aligned}$$

उत्तर

प्रश्न 19. $\int_0^2 \frac{6x+3}{x^2+4} dx.$

हल : ज्ञात है :

$$\begin{aligned} \int_0^2 \frac{6x+3}{x^2+4} dx &= 3 \int_0^2 \frac{2x}{x^2+4} dx + 3 \int_0^2 \frac{dx}{x^2+4} \\ &= \left[3 \log |x^2+4| + 3 \times \frac{1}{2} \tan^{-1} \frac{x}{2} \right]_0^2 \\ &= \left(3 \log 8 + \frac{3}{2} \tan^{-1} \frac{2}{2} \right) - \left(3 \log 4 + \frac{3}{2} \tan^{-1} 0 \right) \\ &= 3 \log \frac{8}{4} + \frac{3}{2} \tan^{-1} 1 \\ &= 3 \log 2 + \frac{3}{2} \cdot \frac{\pi}{4} \\ &= 3 \log 2 + \frac{3\pi}{8}. \end{aligned}$$

उत्तर

प्रश्न 20. $\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx.$

हल : ज्ञात है :

$$\begin{aligned} \int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx &= \int_0^1 xe^x dx + \int_0^1 \sin \frac{\pi x}{4} dx \\ &= [xe^x]_0^1 - \int_0^1 1 \cdot e^x dx + \int_0^1 \sin \frac{\pi x}{4} dx \\ &= [1 \cdot e - 0] - [e^x]_0^1 - \frac{4}{\pi} \left[\cos \pi \frac{x}{4} \right]_0^1 \\ &= e - (e - 1) - \frac{4}{\pi} \left(\cos \frac{\pi}{4} - \cos 0 \right) \\ &= 1 - \frac{4}{\pi} \left(\frac{1}{\sqrt{2}} - 1 \right) \end{aligned}$$

$$= 1 - \frac{2\sqrt{2}}{\pi} + \frac{4}{\pi}$$

$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

उत्तर

प्रश्न 21 एवं 22 में सही उत्तर का चयन कीजिए—

प्रश्न 21. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ बराबर है :

- (A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$
 (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

हल :

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1} x]_1^{\sqrt{3}}$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

अतः विकल्प (D) सही है।

उत्तर

प्रश्न 22. $\int_0^{2/3} \frac{dx}{4+9x^2}$ बराबर है :

- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$
 (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

हल :

$$\int_0^{2/3} \frac{dx}{4+9x^2} = \frac{1}{9} \int_0^{2/3} \frac{dx}{\frac{4}{9} + x^2}$$

$$= \frac{1}{9} \int_0^{2/3} \frac{dx}{\left(\frac{2}{3}\right)^2 + x^2}$$

$$= \frac{1}{9} \cdot \frac{1}{2/3} \left[\tan^{-1} \frac{x}{2/3} \right]_0^{2/3}$$

$$= \frac{1}{9} \times \frac{3}{2} \left[\tan^{-1} \frac{3x}{2} \right]_0^{2/3}$$

$$= \frac{1}{6} \left[\tan^{-1} \frac{3}{2} \times \frac{2}{3} - \tan^{-1} 0 \right]$$

$$= \frac{1}{6} [\tan^{-1}(1) - \tan^{-1}(0)] = \frac{1}{6} \left[\frac{\pi}{4} - 0 \right] = \frac{\pi}{24}$$

अतः विकल्प (C) सही है।

उत्तर

प्रश्नावली 7.10

प्रश्न 1 से 8 तक के प्रश्नों में समाकलनों का मान प्रतिस्थापन का उपयोग करते हुए ज्ञात कीजिए—

प्रश्न 1. $\int_0^1 \frac{x}{x^2 + 1} dx$

हल : दिया है : $\int_0^1 \frac{x}{x^2 + 1} dx = \frac{1}{2} \int_0^1 \frac{2x}{x^2 + 1} dx$

मान लीजिए $x^2 + 1 = t$ हो, तब $\therefore 2x dx = dt$

अतः
$$\begin{aligned} \frac{1}{2} \int_0^1 \frac{2x}{x^2 + 1} dx &= \frac{1}{2} \int_1^2 \frac{dt}{t} = \frac{1}{2} [\log t]_1^2 \\ &= \frac{1}{2} (\log 2 - \log 1) \\ &= \frac{1}{2} \log 2. \end{aligned}$$

उत्तर

प्रश्न 2. $\int_0^{\pi/2} \sqrt{\sin \phi} \cos^5 \phi d\phi$

हल :
$$\begin{aligned} \int_0^{\pi/2} \sqrt{\sin \phi} \cos^5 \phi d\phi &= \int_0^{\pi/2} \sqrt{\sin \phi} \cos^4 \phi \cos \phi d\phi \\ &= \int_0^{\pi/2} \sqrt{\sin \phi} (1 - \sin^2 \phi)^2 \cos \phi d\phi \end{aligned}$$

मान लीजिए $\sin \phi = t$ हो, तो $\cos \phi d\phi = dt$ यदि $\phi = \frac{\pi}{2}, t = 1$ तथा यदि $\phi = 0, t = 0$

$$\begin{aligned} &= \int_0^1 \sqrt{t} (1 - t^2)^2 dt = \int_0^1 \sqrt{t} (1 - 2t^2 + t^4) dt \\ &= \int_0^1 (\sqrt{t} - 2t^{5/2} + t^{9/2}) dt \\ &= \left[\frac{2}{3} t^{3/2} - 2 \cdot \frac{2}{7} t^{7/2} + \frac{2}{11} t^{11/2} \right]_0^1 \\ &= \frac{2}{3} - \frac{4}{7} + \frac{2}{11} = \frac{154 - 132 + 42}{231} = \frac{64}{231}. \end{aligned}$$

उत्तर

प्रश्न 3. $\int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$

हल : $\int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$

मान लीजिए $x = \tan \theta$ हो, तब $\therefore dx = \sec^2 \theta d\theta$

यदि $x = 1, \theta = \frac{\pi}{4}$ तथा यदि $x = 0; \theta = 0$

$$\begin{aligned} &= \int_0^{\pi/4} \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \sec^2 \theta d\theta \\ &= \int_0^{\pi/4} \sin^{-1} (\sin 2\theta) \sec^2 \theta d\theta \end{aligned}$$

$$= \int_0^{\pi/4} 2\theta \sec^2 \theta d\theta = 2 \int_0^{\pi/4} \theta \sec^2 \theta d\theta$$

अब खण्डशः समाकलन करने पर,

$$\begin{aligned} I &= 2[\theta \cdot \tan \theta]_0^{\pi/4} - 2 \int_0^{\pi/4} 1 \cdot \tan \theta d\theta \\ &= 2 \left[\frac{\pi}{4} \tan \frac{\pi}{4} - 0 \right] + 2 \left[\log \cos \theta \right]_0^{\pi/4} \\ &= 2 \left[\frac{\pi}{4} + \log \cos \frac{\pi}{4} - \log \cos 0 \right] \\ &= 2 \left[\frac{\pi}{4} + \log \frac{1}{\sqrt{2}} - \log 1 \right] \\ &= \frac{\pi}{2} - 2 \log \sqrt{2} = \frac{\pi}{2} - 2 \cdot \frac{1}{2} \log 2 = \frac{\pi}{2} - \log 2. \end{aligned}$$

उत्तर

प्रश्न 4. $\int_0^2 x\sqrt{x+2} dx$

हल : $\int_0^2 x\sqrt{x+2} dx$, मान लीजिए $x+2 = t^2$ हो, तब $dx = 2t dt$

$x = t^2 - 2$, यदि $x = 2$, $t = 2$, यदि $x = 0$, $t = \sqrt{2}$

$$\begin{aligned} &= \int_{\sqrt{2}}^2 (t^2 - 2)t \cdot 2t dt = 2 \int_{\sqrt{2}}^2 (t^4 - 2t^2) dt \\ &= 2 \left[\frac{t^5}{5} - 2 \cdot \frac{t^3}{3} \right]_{\sqrt{2}}^2 \\ &= 2 \left[\left(\frac{32}{5} - \frac{2}{3} \times 8 \right) - \left(\frac{(\sqrt{2})^5}{5} - \frac{2}{3} (\sqrt{2})^3 \right) \right] \\ &= 2 \left[\left(\frac{32}{5} - \frac{16}{3} \right) - \left(\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} \right) \right] \\ &= \frac{64}{5} - \frac{32}{3} - \frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3} \\ &= \frac{192 - 160}{15} + \frac{8\sqrt{2}}{15} (5 - 3) \\ &= \frac{32}{15} + \frac{16\sqrt{2}}{15} = \frac{16}{15} (2 + \sqrt{2}) = \frac{16\sqrt{2}}{15} (\sqrt{2} + 1). \end{aligned}$$

उत्तर

प्रश्न 5. $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$

हल : $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$

मान लीजिए $\cos x = t$ हो, तब $\therefore -\sin x \, dx = dt$

जब $x = \frac{\pi}{2}$, $t = 0$, और जब $x = 0$, $t = 1$

$$\begin{aligned} &= -\int_1^0 \frac{dt}{1+t^2} = \int_0^1 \frac{dt}{1+t^2} = \left| \tan^{-1} t \right|_0^1 \\ &= \tan^{-1} 1 - 0 = \frac{\pi}{4}. \end{aligned}$$

उत्तर

प्रश्न 6. $\int_0^2 \frac{dx}{x+4-x^2}$

हल :

$$\begin{aligned} \int_0^2 \frac{dx}{x+4-x^2} &= \int_0^2 \frac{dx}{4 - \left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4}} \\ &= \int_0^2 \frac{dx}{\frac{17}{4} - \left(x - \frac{1}{2}\right)^2} \quad \left[\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} \text{ से} \right] \\ &= \frac{1}{2 \cdot \frac{\sqrt{17}}{2}} \left(\log \frac{\frac{\sqrt{17}}{2} + \left(x - \frac{1}{2}\right)}{\frac{\sqrt{17}}{2} - \left(x - \frac{1}{2}\right)} \right)_0^2 \\ &= \frac{1}{\sqrt{17}} \left[\log \left(\frac{\frac{\sqrt{17}}{2} + \frac{3}{2}}{\frac{\sqrt{17}}{2} - \frac{3}{2}} \right) - \log \left(\frac{\frac{\sqrt{17}}{2} - \frac{1}{2}}{\frac{\sqrt{17}}{2} + \frac{1}{2}} \right) \right] \\ &= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} - \log \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right] \\ &= \frac{1}{\sqrt{17}} \log \left(\frac{\sqrt{17} + 3}{\sqrt{17} - 3} \times \frac{\sqrt{17} + 1}{\sqrt{17} - 1} \right) \\ &= \frac{1}{\sqrt{17}} \log \left(\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}} \right) = \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}} \right) \\ &= \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}} \times \frac{5 + \sqrt{17}}{5 + \sqrt{17}} \right) \\ &= \frac{1}{\sqrt{17}} \log \frac{(5 + \sqrt{17})^2}{25 - 17} \\ &= \frac{1}{\sqrt{17}} \log \left(\frac{42 + 10\sqrt{17}}{8} \right) = \frac{1}{\sqrt{17}} \log \left(\frac{21 + 5\sqrt{17}}{4} \right). \end{aligned}$$

उत्तर

प्रश्न 7. $\int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$

हल : $\int_{-1}^1 \frac{dx}{x^2 + 2x + 5} = \int_{-1}^1 \frac{dx}{(x+1)^2 + 4}$

$\therefore \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

$$= \frac{1}{2} \left(\tan^{-1} \frac{x+1}{2} \right)_{-1}^1$$

$$= \frac{1}{2} [\tan^{-1} 1 - \tan^{-1} 0]$$

$$= \frac{1}{2} \left[\tan^{-1} \left(\tan \frac{\pi}{4} \right) - 0 \right]$$

$$= \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}$$

उत्तर

प्रश्न 8. $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$

हल : $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$

$$= \int_1^2 \left(e^{2x} \cdot \frac{1}{x} - e^{2x} \cdot \frac{1}{2x^2} \right) dx$$

$$= \int_1^2 \frac{e^{2x}}{x} dx - \int_1^2 \frac{e^{2x}}{2x^2} dx$$

प्रथम भाग का खण्डशः समाकलन करने पर,

$$= \left[\frac{1}{x} \cdot \frac{e^{2x}}{2} \right]_1^2 - \int_1^2 \left(-\frac{1}{x^2} \right) \cdot \frac{e^{2x}}{2} dx - \int_1^2 \frac{e^{2x}}{2x^2} dx$$

$$= \left(\frac{e^4}{4} - \frac{e^2}{2} \right) + \int_1^2 \frac{e^{2x}}{2x^2} dx - \int_1^2 \frac{e^{2x}}{2x^2} dx$$

$$= \left(\frac{e^4}{4} - \frac{e^2}{4} \right) = \frac{e^2}{4} (e^2 - 2)$$

उत्तर

प्रश्न 9 एवं 10 में सही उत्तर का चयन कीजिए।

प्रश्न 9. समाकलन $\int_{1/3}^1 \frac{(x - x^3)^{1/3}}{x^4} dx$ का मान है :

- (A) 6 (B) 0 (C) 3 (D) 4

हल : $\int_{1/3}^1 \frac{(x - x^3)^{1/3}}{x^4} dx = \int_{1/3}^1 \frac{(x^3)^{1/3} \left(\frac{x}{x^3} - 1 \right)^{1/3}}{x^4} dx$

$$= \int_{1/3}^1 \frac{\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^3} dx$$

मान लीजिए $\frac{1}{x^2} = t$ हो, तब $\frac{-2}{x^3} dx = dt$

जब $x = 1$ हो, तब $t = 1$ और जब $x = \frac{1}{3}$ हो, तब $t = 9$.

$$\begin{aligned} &= -\frac{1}{2} \int_9^1 (t-1)^{1/3} dt \\ &= -\frac{1}{2} \left[\frac{(t-1)^{4/3}}{4/3} \right]_9^1 \\ &= -\frac{1}{2} \times \frac{3}{4} [(t-1)^{4/3}]_9^1 \\ &= -\frac{3}{8} [0 - (8)^{4/3}] = -\frac{3}{8} [-2^{3 \times \frac{4}{3}}] \\ &= -\frac{3}{8} (-16) = 6 \end{aligned}$$

अतः विकल्प (A) सही है।

उत्तर

प्रश्न 10. यदि $f(x) = \int_0^x t \sin t \, dt$, तब $f'(x)$ है—

- (A) $\cos x + x \sin x$ (B) $x \sin x$
(C) $x \cos x$ (D) $\sin x + x \cos x$

हल : $\therefore f(x) = \int_0^x t \sin t \, dt$

$$\begin{aligned} \therefore \int t \sin t \, dt &= [t(-\cos t) - \int t(-\cos t) \, dt] \\ &= -t \cos t + \sin t \end{aligned}$$

$$\text{अब } \int_0^x t \sin t \, dt = [-t \cos t]_0^x + [\sin t]_0^x = -x \cos x + \sin x$$

$$\therefore f'(x) = -[1(\cos x) - x \sin x] + \cos x = -\cos x + x \sin x + \cos x = x \sin x$$

अतः विकल्प (B) सही है।

उत्तर

प्रश्नावली 7.11

प्रश्न निश्चित समाकलनों के गुणधर्मों का उपयोग करते हुए 1 से 19 तक के प्रश्नों में समाकलनों का मान ज्ञात कीजिए।

प्रश्न 1. $\int_0^{\pi/2} \cos^2 x \, dx$

हल : मान लीजिए

$$I = \int_0^{\pi/2} \cos^2 x \, dx \quad \dots(i)$$

$$= \int_0^{\pi/2} \cos^2 \left(\frac{\pi}{2} - x \right) dx \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \text{ से} \right]$$

$$I = \int_0^{\pi/2} \sin^2 x \, dx \quad \dots(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$\begin{aligned} 2I &= \int_0^{\pi/2} \cos^2 x \, dx + \int_0^{\pi/2} \sin^2 x \, dx \\ &= \int_0^{\pi/2} (\cos^2 x + \sin^2 x) \, dx = \int_0^{\pi/2} 1 \, dx = [x]_0^{\pi/2} \\ &= \frac{\pi}{2} - 0 = \frac{\pi}{2} \\ I &= \frac{\pi}{4} \end{aligned}$$

उत्तर

प्रश्न 2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$

हल : मान लीजिए

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx \quad \dots(i)$$

$$\begin{aligned} &= \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} \, dx \\ &\quad \left[\because \int_0^x f(x) \, dx = \int_a^x f(a-x) \, dx \right] \end{aligned}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \, dx \quad \dots(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \, dx \\ &= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx \\ &= \int_0^{\pi/2} 1 \, dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \end{aligned}$$

$\therefore$

$$I = \frac{\pi}{4} \quad \text{उत्तर}$$

प्रश्न 3. $\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \, dx$

हल : मान लीजिए

$$I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \, dx \quad \dots(i)$$

$$\begin{aligned}
 I &= \int_0^{\pi/2} \frac{\left[\sin\left(\frac{\pi}{2}-x\right)\right]^{3/2}}{\left[\sin\left(\frac{\pi}{2}-x\right)\right]^{3/2} + \left[\cos\left(\frac{\pi}{2}-x\right)\right]^{3/2}} dx \\
 &= \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx \quad \dots(ii)
 \end{aligned}$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$\begin{aligned}
 2I &= \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx + \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx \\
 &= \int_0^{\pi/2} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx \\
 &= \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}
 \end{aligned}$$

$$\therefore I = \frac{\pi}{4} \quad \text{उत्तर}$$

प्रश्न 4. $\int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx$

हल : मान लीजिए

$$I = \int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx \quad \dots(i)$$

$$\begin{aligned}
 I &= \int_0^{\pi/2} \frac{\left[\cos\left(\frac{\pi}{2}-x\right)\right]^5}{\left[\sin\left(\frac{\pi}{2}-x\right)\right]^5 + \left[\cos\left(\frac{\pi}{2}-x\right)\right]^5} dx \\
 &= \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx \quad \dots(ii)
 \end{aligned}$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$\begin{aligned}
 2I &= \int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx + \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx \\
 &= \int_0^{\pi/2} \frac{\cos^5 x + \sin^5 x}{\sin^5 x + \cos^5 x} dx = \int_0^{\pi/2} 1 dx \\
 &= [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}
 \end{aligned}$$

$$\therefore I = \frac{\pi}{4} \quad \text{उत्तर}$$

प्रश्न 5. $\int_{-5}^5 |x+2| dx$

हल :

$$\begin{aligned} \int_{-5}^5 |x+2| dx &= \int_{-5}^{-2} |x+2| dx + \int_{-2}^5 |x+2| dx \\ &= - \int_{-5}^{-2} |x+2| dx + \int_{-2}^5 |x+2| dx \\ &\quad [\because \text{जब } x < -2, |x+2| = -(x+2) \\ &\quad \text{और जब } x > -2, |x+2| = x+2] \\ &= - \left[\frac{x^2}{2} + 2x \right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x \right]_{-2}^5 \\ &= - \left[\left(\frac{(-2)^2}{2} - 4 \right) - \left(\frac{25}{2} - 10 \right) \right] + \left[\left(\frac{25}{2} + 10 \right) - \left(\frac{4}{2} - 4 \right) \right] \\ &= - \left[(-2) - \frac{5}{2} \right] + \left[\frac{45}{2} - (-2) \right] = \frac{9}{2} + \frac{49}{2} = \frac{58}{2} \\ &= 29. \end{aligned}$$

उत्तर

प्रश्न 6. $\int_2^8 |x-5| dx$

हल :

$$\begin{aligned} \int_2^8 |x-5| dx &= \int_2^5 |x-5| dx + \int_5^8 |x-5| dx \\ &\quad \left[\because \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \right] \\ \text{जब } x < 5, |x-5| &= -(x-5) \\ \text{और जब } x > 5, |x-5| &= x-5 \\ \therefore & \\ &= - \left[\frac{x^2}{2} - 5x \right]_2^5 + \left[\frac{x^2}{2} - 5x \right]_5^8 \\ &= - \left[\left(\frac{25}{2} - 25 \right) - \left(\frac{4}{2} - 10 \right) \right] + \left[\left(\frac{64}{2} - 40 \right) - \left(\frac{25}{2} - 25 \right) \right] \\ &= - \left[-\frac{25}{2} - (-8) \right] + \left[-8 - \left(-\frac{25}{2} \right) \right] \\ &= - \left(\frac{-25+16}{2} \right) + \left(\frac{-16+25}{2} \right) = \frac{9}{2} + \frac{9}{2} = 9. \end{aligned}$$

उत्तर

प्रश्न 7. $\int_0^1 x(1-x)^n dx$

हल :

$$\begin{aligned} \int_0^1 x(1-x)^n dx &= \int_0^1 (1-x)[1-(1-x)]^n dx \\ &\quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right] \\ &= \int_0^1 (1-x)x^n dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 (x^n - x^{n+1}) dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^1 \\
&= \frac{1}{n+1} - \frac{1}{n+2} = \frac{n+2-n-1}{(n+1)(n+2)} \\
&= \frac{1}{(n+1)(n+2)}.
\end{aligned}$$

उत्तर

प्रश्न 8. $\int_0^{\pi/4} \log(1 + \tan x) dx$.

हल : मान लीजिए

$$\begin{aligned}
I &= \int_0^{\pi/4} \log(1 + \tan x) dx \\
I &= \int_0^{\pi/4} \log \left[1 + \tan \left(\frac{\pi}{4} - x \right) \right] dx \\
&\quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right] \\
&= \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan x}{1 + \tan x} \right) dx \\
&= \int_0^{\pi/4} \log \left(\frac{1 + \tan x + 1 - \tan x}{1 + \tan x} \right) dx \\
&= \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan x} \right) dx \\
&= \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)] dx \\
&= \log 2 \int_0^{\pi/4} dx - \int_0^{\pi/4} \log(1 + \tan x) dx \\
I &= \log 2 \left[x \right]_0^{\pi/4} - I = \log 2 \cdot \frac{\pi}{4} - I \\
2I &= (\log 2) \frac{\pi}{4} \\
\therefore I &= \frac{\pi}{8} \log 2.
\end{aligned}$$

उत्तर

प्रश्न 9. $\int_0^2 x\sqrt{2-x} dx$

हल : $\int_0^2 x\sqrt{2-x} dx = \int_0^2 (2-x)\sqrt{2-(2-x)} dx$

$$\begin{aligned}
&\quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right] \\
&= \int_0^2 (2-x)\sqrt{x} dx = \int_0^2 (2x^{1/2} - x^{3/2}) dx
\end{aligned}$$

$$\begin{aligned}
&= \left[2 \cdot \frac{2}{3} \cdot x^{3/2} - \frac{2}{5} x^{5/2} \right]_0^2 \\
&= \frac{4}{3} \cdot 2^{3/2} - \frac{2}{5} \cdot 2^{5/2} - 0 \\
&= \frac{4}{3} \cdot 2\sqrt{2} - \frac{2}{5} \cdot 4\sqrt{2} = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = \frac{16\sqrt{2}}{15}.
\end{aligned}$$

उत्तर

प्रश्न 10. $\int_0^{\pi/2} (2 \log \sin x - \log \sin 2x) dx$

हल : मान लीजिए

$$I = \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x) dx \quad \dots(i)$$

$\therefore$

$$I = \int_0^{\pi/2} \left[2 \log \sin \left(\frac{\pi}{2} - x \right) - \log \sin 2 \left(\frac{\pi}{2} - x \right) \right] dx$$

$$I = \int_0^{\pi/2} (2 \log \cos x - \log \sin 2x) dx \quad \dots(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$\begin{aligned}
2I &= \int_0^{\pi/2} \{ [2(\log \sin x + \log \cos x)] - 2 \log \sin 2x \} dx \\
&= \int_0^{\pi/2} 2[\log \sin x \cos x - \log \sin 2x] dx \\
I &= \int_0^{\pi/2} \left[\log \left(\frac{\sin 2x}{2} \right) - \log \sin 2x \right] dx \\
&= \int_0^{\pi/2} (\log \sin 2x - \log 2 - \log \sin 2x) dx \\
&= - \int_0^{\pi/2} \log 2 dx \\
&= - \log 2 \cdot [x]_0^{\pi/2} = - \frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}.
\end{aligned}$$

उत्तर

प्रश्न 11. $\int_{-\pi/2}^{\pi/2} \sin^2 x dx$

हल : मान लीजिए

$$I = \int_{-\pi/2}^{\pi/2} \sin^2 x dx$$

यदि

$$f(-x) = f(x), \text{ तब } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

यहाँ पर

$$\sin^2(-x) = \sin^2 x \quad [\because \sin^2 x \text{ एक सम फलन है}]$$

$\therefore$

$$I = 2 \int_0^{\pi/2} \sin^2 x dx \quad \dots(i)$$

$$= 2 \int_0^{\pi/2} \sin^2 \left(\frac{\pi}{2} - x \right) dx \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$= 2 \int_0^{\pi/2} \cos^2 x dx \quad \dots(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = 2 \int_0^{\pi/2} \sin^2 x dx + 2 \int_0^{\pi/2} \cos^2 x dx$$

$$= 2 \int_0^{\pi/2} (\sin^2 x + \cos^2 x) dx$$

$$= 2 \int_0^{\pi/2} 1 dx = 2[x]_0^{\pi/2}$$

$$= 2 \cdot \frac{\pi}{2} = \pi$$

$$I = \frac{\pi}{2}$$

उत्तर

प्रश्न 12. $\int_0^{\pi} \frac{x dx}{1 + \sin x}$

हल : मान लीजिए

$$I = \int_0^{\pi} \frac{x dx}{1 + \sin x} \quad \dots(i)$$

$$= \int_0^{\pi} \frac{\pi - x}{1 + \sin(\pi - x)} \quad \dots(ii)$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi} \frac{x}{1 + \sin x} dx + \int_0^{\pi} \frac{\pi - x}{1 + \sin x} dx$$

$$= \int_0^{\pi} \frac{x + \pi - x}{1 + \sin x} dx = \pi \int_0^{\pi} \frac{dx}{1 + \sin x}$$

$$= \pi \int_0^{\pi} \frac{1 - \sin x}{1 - \sin^2 x} dx = \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$= \pi \int_0^{\pi} (\sec^2 x - \sec x \tan x) dx$$

$$= \pi \int_0^{\pi} \sec^2 x dx - \pi \int_0^{\pi} \sec x \tan x dx = \pi [\tan x - \sec x]_0^{\pi}$$

$$= \pi [0 - \sec \pi + \sec 0] = \pi [1 + 1] = 2\pi$$

$$I = \pi$$

उत्तर

प्रश्न 13. $\int_{-\pi/2}^{\pi/2} \sin^7 x dx$.

हल : $\int_{-\pi/2}^{\pi/2} \sin^7 x dx$

यहाँ

$$\Rightarrow f(x) = \sin^7 x$$

$$f(-x) = \sin^7(-x) = -\sin^7 x = -f(x)$$

अर्थात् f एक विषम फलन है। $\therefore$

$$I = 0$$

 $\therefore$

$$\int_{-\pi/2}^{\pi/2} \sin^7 x dx = 0.$$

उत्तर

प्रश्न 14. $\int_0^{2\pi} \cos^5 x dx$

हल : माना कि

$$I = \int_0^{2\pi} \cos^5 x dx$$

यदि

$$f(x) = \cos^5 x, f(2\pi - x) = \cos^5 (2\pi - x) \\ = \cos^5 x = f(x)$$

तो

$$I = \int_0^{2\pi} f(x) dx = 2 \int_0^{\pi} f(x) dx$$

 $\therefore$

$$I = \int_0^{2\pi} \cos^5 x dx = 2 \int_0^{\pi} \cos^5 x dx$$

$$\int_0^{2a} f(x) dx = 0 \text{ यदि } f(2a - x) = -f(x)$$

 $\therefore$

$$f(x) = \cos^5 x \\ f(\pi - x) = \cos^5 (\pi - x) = (-\cos x)^5 \\ = -\cos^5 x = -f(x)$$

या

$$f(2a - x) = -f(x)$$

$$\left[\because \int_0^{2a} f(x) dx = 0 \right]$$

 $\therefore$

$$I = 2 \int_0^{\pi} \cos^5 x dx = 0.$$

उत्तर

प्रश्न 15. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$.

हल : मान लीजिए

$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \quad \dots(i)$$

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

 $\therefore$

$$I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \quad \dots(ii)$$

समी. (i) और (ii) को जोड़ने पर

$$2I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx + \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \\ = \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \cos x \cos x} dx = 0$$

 $\therefore$

$$I = 0.$$

उत्तर

प्रश्न 16. $\int_0^{\pi} \log(1 + \cos x) dx$.

हल : मान लीजिए

$$I = \int_0^{\pi} \log(1 + \cos x) dx \quad \dots(i)$$

 $\therefore$

$$I = \int_0^{\pi} \log[1 + \cos(\pi - x)] dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$= \int_0^{\pi} \log(1 - \cos x) dx \quad \dots(ii)$$

समी. (i) और (ii) को जोड़ने पर

$$\begin{aligned}
 2I &= \int_0^\pi \log(1 + \cos x) dx + \int_0^\pi \log(1 - \cos x) dx \\
 &= \int_0^\pi [\log(1 + \cos x) + \log(1 - \cos x)] dx \\
 &= \int_0^\pi \log(1 + \cos x)(1 - \cos x) dx \\
 &= \int_0^\pi \log(1 - \cos^2 x) dx = \int_0^\pi \log \sin^2 x dx \\
 &= 2 \int_0^\pi \log \sin x dx
 \end{aligned}$$

$$\therefore I = \int_0^\pi \log \sin x dx \quad \dots(\text{iii})$$

$$\therefore I = 2 \int_0^{\pi/2} \log \sin x dx = 2I_1 \text{ (माना)} \quad \dots(\text{iv})$$

$$\text{जहाँ } I_1 = \int_0^{\pi/2} \log \sin x dx \quad \dots(\text{v})$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$\begin{aligned}
 I_1 &= \int_0^{\pi/2} \log \sin \left(\frac{\pi}{2} - x \right) dx \\
 &= \int_0^{\pi/2} \log \cos x dx \quad \dots(\text{vi})
 \end{aligned}$$

समी. (v) और (vi) को जोड़ने पर

$$\begin{aligned}
 2I_1 &= \int_0^{\pi/2} \log \sin x dx + \int_0^{\pi/2} \log \cos x dx \\
 &= \int_0^{\pi/2} (\log \sin x + \log \cos x) dx \\
 &= \int_0^{\pi/2} \log \sin x \cos x dx \\
 &= \int_0^{\pi/2} \log \left(\frac{2 \sin x \cos x}{2} \right) dx \\
 &= \int_0^{\pi/2} (\log \sin 2x - \log 2) dx \\
 &= \int_0^{\pi/2} \log \sin 2x dx - (\log 2) \int_0^{\pi/2} 1 dx \\
 &= \int_0^{\pi/2} \log \sin 2x dx - \frac{\pi}{2} \log 2
 \end{aligned}$$

मान लीजिए $2x = t$, हो, तब $2dx = dt$,

$$\begin{aligned}
 \text{अतः } 2I_1 &= \frac{1}{2} \int_0^\pi \log \sin t dt - \frac{\pi}{2} \log 2 \\
 &= \frac{1}{2} \int_0^\pi \log \sin x dx - \frac{\pi}{2} \log 2
 \end{aligned}$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2} \log 2$$

∴

$$2I_1 = I_1 - \frac{\pi}{2} \log 2$$

[समीकरण (v) से]

$$I_1 = -\frac{\pi}{2} \log 2$$

इसका मान समीकरण (iv) में रखने पर

$$I = 2I_1 = -\pi \log 2.$$

उत्तर

प्रश्न 17. $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} \, dx.$

हल : मान लीजिए

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} \, dx \quad \dots(i)$$

∴

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{a-(a-x)}} \, dx$$

$$\left[\because \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right]$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} \, dx \quad \dots(ii)$$

समी. (i) और (ii) को जोड़ने पर

$$2I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} \, dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} \, dx$$

$$= \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} \, dx = \int_0^a 1 \, dx = [x]_0^a = a$$

∴

$$I = \frac{a}{2}.$$

उत्तर

प्रश्न 18. $\int_0^4 |x-1| \, dx.$

हल :

$$\int_0^4 |x-1| \, dx = - \int_0^1 |x-1| \, dx + \int_1^4 |x-1| \, dx$$

$$\left[\because \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \right]$$

$$I = - \int_0^1 (x-1) \, dx + \int_1^4 (x-1) \, dx$$

$$[\because |x-1| = -(x-1) \text{ यदि } x < 1$$

$$\text{तथा } |x-1| = x-1 \text{ यदि } x > 1]$$

$$= - \left[\frac{x^2}{2} - x \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^4$$

$$= -\left[\frac{1}{2}-1\right] + \left(\frac{16}{2}-4\right) - \left(\frac{1}{2}-1\right)$$

$$= \frac{1}{2} + 4 + \frac{1}{2} = 5.$$

उत्तर

प्रश्न 19. दर्शाइए कि $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$, यदि f और g को $f(x) = f(a-x)$ एवं $g(x) + g(a-x) = 4$ के रूप में परिभाषित किया गया है।

हल : $\int_0^a f(x)g(x)dx = \int_0^a f(a-x)g(a-x)dx \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx \right]$

$\therefore$ दिया है $f(a-x) = f(x)$ तथा $g(x) + g(a-x) = 4$

या $g(a-x) = 4 - g(x)$

$$= \int_0^a f(x)[4 - g(x)]dx$$

$$= \int_0^a 4f(x)dx - \int_0^a f(x)g(x)dx$$

$$= 4\int_0^a f(x)dx - \int_0^a f(x)g(x)dx$$

$\therefore 2\int_0^a f(x)g(x)dx = 4\int_0^a f(x)dx$

या $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$

इति सिद्धम्।

प्रश्न 20 एवं 21 में सही उत्तर का चयन कीजिए।

प्रश्न 20. $\int_{-\pi/2}^{\pi/2} (x^3 + x \cos x + \tan^5 x + 1)dx$ का मान है :

- (A) 0 (B) 2 (C) π (D) 1

हल : मान लीजिए

$$I = \int_{-\pi/2}^{\pi/2} (x^3 + x \cos x + \tan^5 x + 1)dx$$

$$= \int_{-\pi/2}^{\pi/2} (x^3 + x \cos x + \tan^5 x)dx + \int_{-\pi/2}^{\pi/2} 1dx$$

पुनः माना

$$f(x) = x^3 + x \cos x + \tan^5 x$$

$\therefore$

$$f(-x) = (-x)^3 + (-x) \cos(-x) + \tan^5(-x)$$

$$= -x^3 - x \cos x - \tan^5 x$$

$$= -(x^3 + x \cos x + \tan^5 x)$$

$$= -f(x)$$

$\therefore$

$$I = \int_{-\pi/2}^{\pi/2} f(x) - \int_{-\pi/2}^{\pi/2} f(x) + [x]_{-\pi/2}^{\pi/2}$$

$$= 0 + \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right]$$

$$\left[\because \int_{-a}^a f(x)dx = 0, \text{ यदि } f \text{ एक विषम फलन है} \right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \frac{2\pi}{2} = \pi$$

अतः विकल्प (C) सही है।

उत्तर

प्रश्न 21. $\int_a^{\pi/2} \log \left(\frac{4+3\sin x}{4+3\cos x} \right) dx$ का मान है :

- (A) 2 (B) $\frac{3}{4}$ (C) 0 (D) -2

हल : मान लीजिए

$$I = \int_0^{\pi/2} \log \left(\frac{4+3\sin x}{4+3\cos x} \right) dx \quad \dots(i)$$

$$I = \int_0^{\pi/2} \log \left(\frac{4+3\sin \left(\frac{\pi}{2} - x \right)}{4+3\cos \left(\frac{\pi}{2} - x \right)} \right) dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$I = \int_0^{\pi/2} \log \left(\frac{4+3\cos x}{4+3\sin x} \right) dx \quad \dots(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \log \left(\frac{4+3\sin x}{4+3\cos x} \right) + \log \left(\frac{4+3\cos x}{4+3\sin x} \right) dx$$

$$= \int_0^{\pi/2} \log \left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x} \right) dx$$

$$= \int_0^{\pi/2} \log 1 dx = 0$$

$$I = 0$$

या

अतः विकल्प (C) सही है।

उत्तर

अध्याय 7 पर विविध प्रश्नावली

प्रश्न 1 से 24 तक के प्रश्नों के फलनों का समाकलन कीजिए—

प्रश्न 1. $\frac{1}{x-x^3}$.

हल :

$$\frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1+x)(1-x)}$$

अब

$$\frac{1}{x(1+x)(1-x)} = \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x}$$

$\therefore$

$$1 = A(1-x^2) + Bx(1-x) + Cx(1+x)$$

$x = 0$ रखने पर,

$$1 = A \text{ या } A = 1$$

$x = -1$ रखने पर,

$$1 = B(-1)(1+1) = -2B \text{ या } B = -\frac{1}{2}$$

$x = 1$ रखने पर,

$$1 = C. 1. (1 + 1) = 2C \text{ या } C = \frac{1}{2}$$

$$\therefore \frac{1}{x-x^3} = \frac{1}{x} - \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

$$\begin{aligned} \text{अतः} \quad \int \frac{1}{x-x^3} dx &= \int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{1+x} dx + \frac{1}{2} \int \frac{1}{1-x} dx \\ &= \log |x| - \frac{1}{2} \log |1+x| - \frac{1}{2} \log |1-x| + C \\ &= \frac{1}{2} \log |x|^2 - \frac{1}{2} \log |1-x^2| + C \\ &= \frac{1}{2} \log \left| \frac{x^2}{1-x^2} \right| + C. \end{aligned}$$

उत्तर

प्रश्न 2. $\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$

$$\begin{aligned} \text{हल :} \quad \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx &= \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \times \left(\frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}} \right) dx \\ &= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{x+a-x-b} dx \\ &= \frac{1}{a-b} \int [(x+a)^{1/2} - (x+b)^{1/2}] dx \\ &= \frac{1}{a-b} \left[\frac{(x+a)^{3/2}}{\frac{3}{2}} - \frac{(x+b)^{3/2}}{\frac{3}{2}} \right] + C \\ &= \frac{2}{3(a-b)} [(x+a)^{3/2} - (x+b)^{3/2}] + C. \end{aligned}$$

उत्तर

प्रश्न 3. $\frac{1}{x\sqrt{ax-x^2}}$

हल : $\int \frac{1}{x\sqrt{ax-x^2}} dx$

$x = \frac{a}{t}$ रखने पर तब $dx = -\frac{a}{t^2} dt$

$$\begin{aligned} &= \int \frac{-\frac{a}{t^2}}{\frac{a}{t} \sqrt{\frac{a^2}{t} - \frac{a^2}{t^2}}} dt = -\frac{1}{a} \int \frac{\frac{1}{t}}{\sqrt{t-1}} dt \\ &= -\frac{1}{a} \int \frac{1}{\sqrt{t-1}} dt = -\frac{1}{a} \int (t-1)^{-1/2} dt \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{a} \cdot \frac{(t-1)^{1/2}}{\frac{1}{2}} + C = -\frac{2}{a} \sqrt{t-1} + C \\
&= -\frac{2}{a} \sqrt{\frac{a}{x} - 1} + C = -\frac{2}{a} \sqrt{\frac{a-x}{x}} + C.
\end{aligned}$$

उत्तर

प्रश्न 4. $\int \frac{dx}{x^2(x^4+1)^{3/4}}$

हल :

$$\begin{aligned}
&\int \frac{dx}{x^2(x^4+1)^{3/4}} \\
&= \int \frac{dx}{x^2(x^4)^{3/4} \left(1 + \frac{1}{x^4}\right)^{3/4}} \\
&= \int \frac{dx}{x^2 x^3 \left(1 + \frac{1}{x^4}\right)^{3/4}} \\
&= \int \left(1 + \frac{1}{x^4}\right)^{-3/4} x^{-5} dx
\end{aligned}$$

मान लीजिए

$$1 + \frac{1}{x^4} = t$$

या

$$1 + x^{-4} = t$$

तब

$$-4x^{-5} dx = dt$$

 $\Rightarrow$

$$\begin{aligned}
x^{-5} dx &= -\frac{1}{4} dt \\
&= -\frac{1}{4} \int t^{-3/4} dt \\
&= -\frac{1}{4} \cdot \frac{\left(1 + \frac{1}{x^4}\right)^{1-\frac{3}{4}}}{1-\frac{3}{4}} + C \\
&= -\left(1 + \frac{1}{x^4}\right)^{1/4} + C.
\end{aligned}$$

उत्तर

प्रश्न 5. $\frac{1}{x^{1/2} + x^{1/3}}$

हल : $\int \frac{1}{x^{1/2} + x^{1/3}} dx = \frac{1}{x^{1/3}(1+x^{1/6})}$

अब

$$x^{1/6} = t \text{ रखने पर या } x = t^6 \text{ हो, तब } dx = 6t^5 dt$$

$$\begin{aligned}
&= \int \frac{6t^5}{t^3 + t^2} dt = 6 \int \frac{t^5}{t^2(t+1)} dt = 6 \int \frac{t^3}{t+1} dt \\
&= 6 \int \frac{t^3 + 1 - 1}{t+1} dt = 6 \int \left(\frac{t^3 + 1}{t+1} - \frac{1}{t+1} \right) dt \\
&= 6 \int \left(\frac{(t+1)(t^2 - t + 1)}{t+1} - \frac{1}{t+1} \right) dt \\
&= 6 \int \left(t^2 - t + 1 - \frac{1}{t+1} \right) dt \\
&= 6 \left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1| \right) + C \\
&= 6 \left[\frac{1}{3}(x^{1/6})^3 - \frac{1}{2}(x^{1/6})^2 + x^{1/6} - \log(x^{1/6} + 1) \right] + C \\
&= 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log(x^{1/6} + 1) + C. \quad \text{उत्तर}
\end{aligned}$$

प्रश्न 6. $\frac{5x}{(x+1)(x^2+9)}$.

हल : आंशिक भिन्न के प्रयोग से,

$$\frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$\begin{aligned}
\therefore 5x &= A(x^2+9) + (Bx+C)(x+1) \\
&= A(x^2+9) + B(x^2+x) + C(x+1)
\end{aligned}$$

$x = -1$ लेने पर,

$$-5 = A \times 10 \text{ या } A = -\frac{1}{2}$$

x^2 के गुणांकों की तुलना करने पर

$$0 = A + B \text{ या } B = -A \text{ या } B = \frac{1}{2}$$

अचर राशि की तुलना करने पर

$$0 = 9A + C \text{ या } C = -9A = \frac{9}{2}$$

$$\therefore \frac{5x}{(x+1)(x^2+9)} = -\frac{1}{2(x+1)} + \frac{\frac{1}{2}x + \frac{9}{2}}{x^2+9}$$

$$= -\frac{1}{2(x+1)} + \frac{1}{2} \left(\frac{x+9}{x^2+9} \right)$$

$$\begin{aligned}
\therefore \int \frac{5x}{(x+1)(x^2+9)} dx &= -\frac{1}{2} \int \frac{1}{x+1} dx + \frac{1}{2} \int \frac{x+9}{x^2+9} dx \\
&= -\frac{1}{2} \log|x+1| + \frac{1}{4} \int \frac{2x}{x^2+9} dx + \frac{9}{2} \int \frac{1}{x^2+9} dx + C
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \log |x+1| + \frac{1}{4} \log |x^2+9| + \frac{9}{2} \times \frac{1}{3} \tan^{-1} \frac{x}{3} + C \\
&= -\frac{1}{2} \log |x+1| + \frac{1}{4} \log |x^2+9| + \frac{3}{2} \tan^{-1} \frac{x}{3} + C. \quad \text{उत्तर}
\end{aligned}$$

प्रश्न 7. $\frac{\sin x}{\sin(x-a)}$.

हल : $\int \frac{\sin x}{\sin(x-a)} dx = \int \frac{\sin(x-a+a)}{\sin(x-a)} dx$

$$\begin{aligned}
&= \int \frac{\sin(x-a) \cos a + \cos(x-a) \sin a}{\sin(x-a)} dx \\
&= \int [\cos a + \cot(x-a) \sin a] dx \\
&= x \cos a + \sin a \log |\sin(x-a)| + C \\
&= \sin a \log |\sin(x-a)| + x \cos a + C.
\end{aligned}$$

उत्तर

प्रश्न 8. $\frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}}$.

हल : $\int \frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}} dx$

$$\begin{aligned}
&= \int \frac{e^{\log x^5} - e^{\log x^4}}{e^{\log x^3} - e^{\log x^2}} dx \\
&= \int \frac{x^5 - x^4}{x^3 - x^2} dx \\
&= \int \frac{x^4(x-1)}{x^2(x-1)} dx \\
&= \int x^2 dx = \frac{x^3}{3} + C.
\end{aligned}$$

उत्तर

प्रश्न 9. $\frac{\cos x}{\sqrt{4-\sin^2 x}}$.

हल : $\int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx$

मान लीजिए $\sin x = t$ हो, तब $\cos x dx = dt$

$$= \int \frac{dt}{\sqrt{4-t^2}} = \sin^{-1} \frac{t}{2} + C = \sin^{-1} \left(\frac{\sin x}{2} \right) + C.$$

उत्तर

प्रश्न 10. $\frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x}$

हल : $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$

यहाँ $\sin^8 x - \cos^8 x$

$$\begin{aligned} &= (\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x) \\ &= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x) \\ &= (\sin^2 x - \cos^2 x)\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\} \\ &\quad [\because a^2 + b^2 = (a+b)^2 - 2ab] \\ &= (\sin^2 - \cos^2 x)[1 - 2\sin^2 x \cos^2 x] \end{aligned}$$

अतः $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x - \cos^2 x)(1 - 2\sin^2 x \cos^2 x)}{1 - 2\sin^2 x \cos^2 x} dx$

$$= \int (\sin^2 x - \cos^2 x) dx = -\int \cos 2x dx$$

$$= -\frac{\sin 2x}{2} + C = -\frac{1}{2} \sin 2x + C.$$

उत्तर

प्रश्न 11. $\frac{1}{\cos(x+a)\cos(x+b)}$

हल : $\int \frac{1}{\cos(x+a)\cos(x+b)} dx = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} dx$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x+a-x-b)}{\cos(x+a)\cos(x+b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(\overline{x+a-x+b})}{\cos(x+a)\cos(x+b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)} dx$$

$$= \frac{1}{\sin(a-b)} \int [(\tan(x+a) - \tan(x+b))] dx$$

$$= \frac{1}{\sin(a-b)} [-\log|\cos(x+a)| + \log|\cos(x+b)|] + C$$

$$= -\frac{1}{\sin(a-b)} \log \left| \frac{\cos(x+a)}{\cos(x+b)} \right| + C$$

$$= \frac{1}{\sin(a-b)} \log \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C.$$

उत्तर

प्रश्न 12. $\frac{x^3}{\sqrt{1-x^8}}$

हल :

$$\int \frac{x^3}{\sqrt{1-x^8}} dx = \int \frac{x^3 dx}{\sqrt{1-(x^4)^2}}$$

मान लीजिए $x^4 = t$ रखने पर, $4x^3 dx = dt$

$$\begin{aligned} &= \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{4} \sin^{-1} t + C \\ &= \frac{1}{4} \sin^{-1} x^4 + C. \end{aligned}$$

उत्तर

प्रश्न 13. $\frac{e^x}{(1+e^x)(2+e^x)}$

हल : $\int \frac{e^x}{(1+e^x)(2+e^x)} dx$

यहाँ $e^x = t$ रखने पर, $e^x dx = dt$

$$= \int \frac{dt}{(1+t)(2+t)}$$

अब

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$$

$\therefore$
 $t = -1$ लेने पर,

$$1 = A(2+t) + B(1+t)$$

तथा $t = -2$ लेने पर,

$$1 = A \times 1 \text{ या } A = 1$$

$$1 = B(1-2) \text{ या } B = -1$$

$$= \int \frac{A}{1+t} dt + \int \frac{B}{2+t} dt$$

$$= \int \frac{1}{1+t} dt - \int \frac{1}{2+t} dt$$

$$= \log |1+t| - \log |2+t| + C$$

$$= \log \left(\frac{1+t}{2+t} \right) + C = \log \left| \frac{1+e^x}{2+e^x} \right| + C.$$

उत्तर

प्रश्न 14. $\frac{1}{(x^2+1)(x^2+4)}$

हल : $\int \frac{1}{(x^2+1)(x^2+4)}$

$x^2 = y$ रखने पर, $\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$

$\therefore$ $1 = A(y+4) + B(y+1)$

$y = -1$ रखने पर, $1 = A(-1+4) = 3A$ या $A = \frac{1}{3}$

तथा $y = -4$ रखने पर, $1 = B(-4+1) = -3B$ या $B = -\frac{1}{3}$

अर्थात् $\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(y+1)} - \frac{1}{3(y+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$

$$\begin{aligned}
&= \int \frac{1}{3(x^2+1)} dx - \frac{1}{3} \int \frac{1}{x^2+4} dx \\
&= \frac{1}{3} \tan^{-1} x - \frac{1}{3} \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} + C \\
&= \frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C.
\end{aligned}$$

उत्तर

प्रश्न 15. $\cos^3 x e^{\log \sin x}$.

हल : $\int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x \sin x dx$

मान लीजिए $\cos x = t$ हो, तब, $-\sin x dx = dt$

$$[\because e^{\log x} = x]$$

$$\begin{aligned}
&= \int t^3 (-dt) = -\int t^3 dt = -\frac{t^4}{4} + C \\
&= -\frac{\cos^4 x}{4} + C = -\frac{1}{4} \cos^4 x + C.
\end{aligned}$$

उत्तर

प्रश्न 16. $e^{3 \log x} (x^4 + 1)^{-1}$.

हल : $\int e^{3 \log x} (x^4 + 1)^{-1} dx$

$$= \int \frac{e^{\log x^3}}{x^4 + 1} dx = \int \frac{x^3}{x^4 + 1} dx$$

$$[\because e^{\log x} = x]$$

मान लीजिए $x^4 + 1 = t$ हो, तब, $4x^3 dx = dt$

$$\begin{aligned}
&= \frac{1}{4} \int \frac{dt}{t} = \frac{1}{4} \log |t| + C \\
&= \frac{1}{4} \log |x^4 + 1| + C.
\end{aligned}$$

उत्तर

प्रश्न 17. $f'(ax+b) [f(ax+b)]^n$.

हल : $\int [f(ax+b)]^n f'(ax+b) dx$

अब $(ax+b) = t$ रखने पर

$$\therefore f'(ax+b) \cdot adx = dt$$

$$\begin{aligned}
&= \int t^n \cdot \frac{1}{a} dt = \frac{1}{a} \cdot \frac{t^{n+1}}{n+1} + C \\
&= \frac{[f(ax+b)]^{n+1}}{a(n+1)} + C.
\end{aligned}$$

उत्तर

प्रश्न 18. $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$.

हल : $\int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx$.

अब

$$\begin{aligned}
\sin^3 x \sin(x+\alpha) &= \sin^3 x (\sin x \cos \alpha + \cos x \sin \alpha) \\
&= \sin^4 x (\cos \alpha + \cot x \sin \alpha)
\end{aligned}$$

$$= \int \frac{1}{\sqrt{\sin^4 x (\cos \alpha + \cot x \sin \alpha)}} dx$$

$$= \int \frac{1}{\sin^2 x \sqrt{\cos \alpha + \cot x \sin \alpha}} dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}} dx$$

मान लीजिए $\cos \alpha + \cot x \sin \alpha = t$ हो, तब $\therefore -\operatorname{cosec}^2 x \sin \alpha dx = dt$

$$= -\frac{1}{\sin \alpha} \int \frac{dt}{\sqrt{t}} = -\frac{1}{\sin \alpha} \int t^{-1/2} dt$$

$$= -\frac{1}{\sin \alpha} \frac{t^{1/2}}{\frac{1}{2}} + C$$

$$= -\frac{2}{\sin \alpha} \sqrt{\cos \alpha + \cot x \sin \alpha} + C$$

$$= -\frac{2}{\sin \alpha} \sqrt{\cos \alpha + \frac{\cos x}{\sin x} \sin \alpha} + C$$

$$= -\frac{2}{\sin \alpha} \sqrt{\frac{\sin x \cos \alpha + \cos x \sin \alpha}{\sin x}} + C$$

$$= -\frac{2}{\sin \alpha} \sqrt{\frac{\sin (x + \alpha)}{\sin x}} + C.$$

उत्तर

प्रश्न 19. $\frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}}, (x \in [0, 1])$.

हल : $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$

चूँकि प्रतिलोम वृत्तीय फलन से,

$$\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2}$$

$$\therefore \cos^{-1} \sqrt{x} = \frac{\pi}{2} - \sin^{-1} \sqrt{x}$$

$$= \int \frac{\sin^{-1} \sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1} \sqrt{x} \right)}{\frac{\pi}{2}} dx$$

$$= \frac{2}{\pi} \int 2 \sin^{-1} \sqrt{x} dx - \int dx$$

$$= \frac{4}{\pi} \int \sin^{-1} \sqrt{x} dx - x$$

मान लीजिए $x = t^2$ हो, तब $dx = 2t dt$

$$= \frac{4}{\pi} \int \sin^{-1} t \cdot 2t dt - x$$

$$= -x + \frac{8}{\pi} \int (\sin^{-1} t) t dt$$

खण्डशः समाकलन करने पर,

$$= -x + \frac{8}{\pi} \left[(\sin^{-1} t) \frac{t^2}{2} - \int \frac{1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt \right]$$

$$= -x + \frac{4}{\pi} t^2 \sin^{-1} t - \frac{4}{\pi} \int \frac{t^2}{\sqrt{1-t^2}} dt$$

$$= -x + \frac{4}{\pi} t^2 \sin^{-1} t + \frac{4}{\pi} \int \frac{1-t^2-1}{\sqrt{1-t^2}} dt$$

$$= -x + \frac{4}{\pi} t^2 \sin^{-1} t + \frac{4}{\pi} \int \left(\sqrt{1-t^2} - \frac{1}{\sqrt{1-t^2}} \right) dt$$

$$= -x + \frac{4}{\pi} t^2 \sin^{-1} t + \frac{4}{\pi} \left[\frac{t\sqrt{1-t^2}}{2} + \frac{1}{2} \sin^{-1} t - \sin^{-1} t \right] + C$$

$$= -x + \frac{4}{\pi} t^2 \sin^{-1} t + \frac{2}{\pi} t \sqrt{1-t^2} - \frac{2}{\pi} \sin^{-1} t + C$$

$$= -x + \frac{4}{\pi} x \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x} \sqrt{1-x} - \frac{2}{\pi} \sin^{-1} \sqrt{x} + C$$

$$= -x + \frac{2}{\pi} (2x-1) \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x} \sqrt{1-x} + C$$

$$= \frac{2(2x-1)}{\pi} \sin^{-1} \sqrt{x} + \frac{2\sqrt{x-x^2}}{\pi} - x + C.$$

उत्तर

प्रश्न 20. $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$.

हल : $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$

मान लीजिए $\sqrt{x} = \cos t$ या $x = \cos^2 t$ रखने पर

$\therefore$

$$dx = -2 \cos t \sin t dt$$

$$= \int \sqrt{\frac{1-\cos t}{1+\cos t}} (-2 \cos t \sin t) dt$$

$$= -2 \int \sqrt{\frac{2 \sin^2 \frac{t}{2}}{2 \cos^2 \frac{t}{2}}} \times \cos t \sin t dt$$

$$\begin{aligned}
&= -2 \int \frac{\sin \frac{t}{2}}{\cos \frac{t}{2}} \left(2 \sin \frac{t}{2} \cos \frac{t}{2} \cos t \right) dt \\
&\quad \left[\because \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \right] \\
&= -4 \int \sin^2 \frac{t}{2} \cos t \, dt = -4 \int \frac{1 - \cos t}{2} \cos t \, dt \\
&= -2 \int (\cos t - \cos^2 t) \, dt \\
&= -2 \int \left[\cos t - \frac{1 + \cos 2t}{2} \right] dt \\
&= -2 \sin t + \left(t + \frac{\sin 2t}{2} \right) + C \\
&= t + \sin t \cos t - 2 \sin t + C \\
\text{अब पुनः } \cos t = \sqrt{x} \text{ रखने पर } \therefore \sin t = \sqrt{1-x}, t = \cos^{-1} \sqrt{x}, \text{ के मान रखने पर} \\
&= \cos^{-1} \sqrt{x} - 2\sqrt{1-x} + \sqrt{x}\sqrt{1-x} + C \\
&= -2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x-x^2} + C.
\end{aligned}$$

उत्तर

प्रश्न 21. $\frac{2 + \sin 2x}{1 + \cos 2x} e^x$.

$$\begin{aligned}
\text{हल : } \int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx &= \int \frac{2 + 2 \sin x \cos x}{2 \cos^2 x} e^x dx \\
&= \int \frac{1 + \sin x \cos x}{\cos^2 x} e^x dx \\
&= \int e^x (\tan x + \sec^2 x) dx
\end{aligned}$$

अब $e^x \tan x = t$ रखने पर

$$\begin{aligned}
(e^x \sec^2 x + e^x \tan x) dx &= dt \text{ या } e^x (\tan x + \sec^2 x) dx = dt \\
&= \int dt = t + C = e^x \tan x + C.
\end{aligned}$$

उत्तर

प्रश्न 22. $\frac{x^2 + x + 1}{(x+1)^2(x+2)}$

$$\text{हल : } \int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx$$

$$\begin{aligned}
\text{अब } \frac{x^2 + x + 1}{(x+1)^2(x+2)} &= \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\
\therefore x^2 + x + 1 &= A(x+1)^2 + B(x+2)(x+1) + C(x+2) \\
&= A(x^2 + 2x + 1) + B(x^2 + 3x + 2) + C(x+2) \\
x = -2 \text{ लेने पर,}
\end{aligned}$$

$$4 - 2 + 1 = A(-1)^2 \text{ या } 3 = A \text{ या } A = 3$$

तथा $x = -1$ लेने पर,

$$1 - 1 + 1 = C(-1 + 2) = C \text{ या } C = 1$$

x^2 के गुणांकों की तुलना करने पर,

$$1 = A + B \text{ या } B = 1 - A = 1 - 3 = -2$$

$$\therefore \frac{x^2 + x + 1}{(x+2)(x+1)^2} = \frac{3}{x+2} - \frac{2}{x+1} + \frac{1}{(x+1)^2}$$

अब

$$\begin{aligned} \int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx &= 3 \int \frac{1}{x+2} dx - 2 \int \frac{1}{x+1} dx + \int \frac{1}{(x+1)^2} dx \\ &= 3 \log |x+2| - 2 \log |x+1| + \int (x+1)^{-2} dx \\ &= 3 \log |x+2| - 2 \log |x+1| + \frac{(x+1)^{-1}}{-1} + C \\ &= 3 \log |x+2| - 2 \log |x+1| - \frac{1}{x+1} + C \\ &= -2 \log |x+1| - \frac{1}{x+1} + 3 \log |x+2| + C. \end{aligned}$$

उत्तर

प्रश्न 23. $\tan^{-1} \sqrt{\frac{1-x}{1+x}}$.

हल : माना कि

$$I = \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$$

मान लीजिए $x = \cos \theta$ हो, तब, $dx = -\sin \theta d\theta$

$$\begin{aligned} &= -\int \tan^{-1} \left(\sqrt{\frac{1-\cos \theta}{1+\cos \theta}} \right) \sin \theta d\theta \\ &= -\int \tan^{-1} \left(\sqrt{\frac{2\sin^2 \frac{\theta}{2}}{2\cos^2 \frac{\theta}{2}}} \right) \sin \theta d\theta \\ &= -\int \tan^{-1} \left(\tan \frac{\theta}{2} \right) \sin \theta d\theta \\ &= -\int \frac{\theta}{2} \sin \theta d\theta = -\frac{1}{2} \int \theta \sin \theta d\theta \end{aligned}$$

खण्डशः समाकलन करने पर,

$$\begin{aligned} &= -\frac{1}{2} [\theta(-\cos \theta) - \int 1.(-\cos \theta) d\theta] \\ &= \frac{1}{2} \theta \cos \theta - \frac{1}{2} \sin \theta + C \end{aligned}$$

$$\begin{aligned}
\text{पुनः } x = \cos \theta \text{ रखने पर } \theta &= \cos^{-1} x, \sin \theta = \sqrt{1-x^2} \\
&= \frac{1}{2} x \cos^{-1} x - \frac{1}{2} \sqrt{1-x^2} + C \\
&= \frac{1}{2} (x \cos^{-1} x - \sqrt{1-x^2}) + C.
\end{aligned}$$

उत्तर

प्रश्न 24. $\frac{\sqrt{x^2+1} [\log(x^2+1) - 2 \log x]}{x^4}$.

हल : $\int \frac{\sqrt{x^2+1} [\log(x^2+1) - 2 \log x]}{x^4} dx$

$$\begin{aligned}
&= \int \frac{\sqrt{x^2+1} \log \left(\frac{x^2+1}{x^2} \right)}{x^4} dx \\
&= \int \frac{x \left(\sqrt{1+\frac{1}{x^2}} \right) \log \left(1 + \frac{1}{x^2} \right)}{x^4} dx \\
&= \int \frac{\sqrt{1+\frac{1}{x^2}} \log \left(1 + \frac{1}{x^2} \right)}{x^3} dx
\end{aligned}$$

मान लीजिए

$$\begin{aligned}
1 + \frac{1}{x^2} &= t \text{ हो, तब } \frac{-2}{x^3} dx = dt \\
&= \int \frac{\sqrt{t} \log t}{-2} dt \\
&= -\frac{1}{2} \int \sqrt{t} \log t \, dt \\
&= -\frac{1}{2} \log t \times \frac{t^{3/2}}{\frac{3}{2}} - \int \frac{t^{3/2}}{\frac{3}{2}} \times \frac{1}{t} dt \\
&= -\frac{1}{3} \left[t^{3/2} \log t - \int \sqrt{t} \, dt \right] \\
&= -\frac{1}{3} \left[t^{3/2} \log t - \frac{t^{3/2}}{\frac{3}{2}} \right] + C \\
&= -\frac{1}{3} t^{3/2} \left[\log t - \frac{2}{3} \right] + C \\
&= -\frac{1}{3} \left(1 + \frac{1}{x^2} \right)^{3/2} \left[\log \left(1 + \frac{1}{x^2} \right) - \frac{2}{3} \right] + C.
\end{aligned}$$

उत्तर

प्रश्न 25 से 33 तक के प्रश्नों में निश्चित समाकलनों का मान ज्ञात कीजिए।

प्रश्न 25. $\int_{\pi/2}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx.$

हल : $\int_{\pi/2}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$

$$= \int_{\pi/2}^{\pi} e^x \left(\frac{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) dx$$

$$= - \int_{\pi/2}^{\pi} e^x \left(\cot \frac{x}{2} - \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \right) dx$$

मान लीजिए $e^x \cot \frac{x}{2} = t$ हो, तो $\left(e^x \cot \frac{x}{2} - \frac{1}{2} e^x \operatorname{cosec}^2 \frac{x}{2} \right) dx = dt$

या $e^x \left(\cot \frac{x}{2} - \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \right) dx = dt$

$$= - \int dt = -t + C = -e^x \cot \frac{x}{2} + C$$

$$\therefore \int_{\pi/2}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx = - \int_{\pi/2}^{\pi} dt = [-t]_{\pi/2}^{\pi} + C = \left[-e^x \cot \frac{x}{2} \right]_{\pi/2}^{\pi}$$

$$= -e^{\pi} \cot \frac{\pi}{2} + e^{\pi/2} \cot \frac{\pi}{4}$$

$$= e^{\pi} (-0) + e^{\pi/2} \cdot 1$$

$$= e^{\pi/2}.$$

उत्तर

प्रश्न 26. $\int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx.$

हल : $\int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

अंश व हर को $\cos^4 x$ से भाग देने पर

$$\int_0^{\pi/4} \frac{\tan x \sec^2 x dx}{1 + \tan^4 x}$$

मान लीजिए $\tan^2 x = t$ हो, तब

$$2 \tan x \sec^2 x dx = dt$$

अब $x = 0$ हो, तो $t = 0$ और जब $x = \pi/4$ हो, तब $t = 1$.

$$= \frac{1}{2} \int_0^1 \frac{dt}{1 + t^2}$$

$$= \frac{1}{2} \left[\tan^{-1} t \right]_0^1 = \frac{1}{2} \tan^{-1} 1 = \frac{\pi}{8}.$$

उत्तर

प्रश्न 27. $\int_0^{\pi/2} \frac{\cos^2 x dx}{\cos^2 x + 4 \sin^2 x}$.

हल :
$$\begin{aligned} \int_0^{\pi/2} \frac{\cos^2 x dx}{\cos^2 x + 4 \sin^2 x} &= \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4 - 4 \cos^2 x} dx \\ &= \int_0^{\pi/2} \frac{\cos^2 x}{4 - 3 \cos^2 x} dx \\ &= \frac{1}{3} \int_0^{\pi/2} \frac{3 \cos^2 x}{4 - 3 \cos^2 x} dx \\ &= \frac{-1}{3} \int_0^{\pi/2} \frac{4 - 3 \cos^2 x - 4}{4 - 3 \cos^2 x} dx \\ &= \frac{-1}{3} \int_0^{\pi/2} \frac{4 - 3 \cos^2 x}{4 - 3 \cos^2 x} dx - \frac{4}{3} \int_0^{\pi/2} \frac{dx}{4 - 3 \cos^2 x} \\ &= -\frac{1}{3} [x]_0^{\pi/2} - \frac{4}{3} \int_0^{\pi/2} \frac{dx}{4 - 3 \cos^2 x} \\ &= -\frac{\pi}{6} + \frac{4}{3} \int_0^{\pi/2} \frac{\sec^2 x}{4 \sec^2 x - 3} dx \\ &\quad [\cos^2 \text{ से अंश व हर से भाग करने पर}] \\ &= -\frac{\pi}{6} + \frac{4}{3} \int_0^{\pi/2} \frac{\sec^2 x}{4 (1 + \tan^2 x) - 3} dx \\ &= -\frac{\pi}{6} + \frac{4}{3} \int_0^{\pi/2} \frac{\sec^2 x}{1 + 4 \tan^2 x} dx \end{aligned}$$

पुनः $\tan x = t$ रखने पर, $\sec^2 x dx = dt$, जब $x = \frac{\pi}{2}$ तो $t = \tan \frac{\pi}{2} = \infty$

जब $x = 0$ तो $t = 0$

$$\begin{aligned} &= -\frac{\pi}{6} + \frac{4}{3} \int_0^{\infty} \frac{1}{1 + 4t^2} dt = -\frac{\pi}{6} + \frac{4}{3} \cdot \frac{1}{2} [\tan^{-1} 2t]_0^{\infty} \\ &= -\frac{\pi}{6} + \frac{2}{3} \cdot \frac{\pi}{2} = -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6} \end{aligned}$$

उत्तर

प्रश्न 28. $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$.

हल : $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$$\begin{aligned} &= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - [1 - \sin 2x]}} dx \\ &= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - [\sin x - \cos x]^2}} dx \end{aligned}$$

मान लीजिए $\sin x - \cos x = t$ हो, तब, $(\cos x + \sin x) dx = dt$

$$\text{जब } x = \frac{\pi}{3} \text{ तो } t = \sin \frac{\pi}{3} - \cos \frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

$$\text{तथा जब } x = \frac{\pi}{6} \text{ तो } t = \sin \frac{\pi}{6} - \cos \frac{\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}-1}{2}$$

$$\begin{aligned} \therefore I &= \int_{\frac{\sqrt{3}-1}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t \right]_{\frac{\sqrt{3}-1}{2}}^{\frac{\sqrt{3}-1}{2}} \\ &= - \left[\sin^{-1} \frac{1-\sqrt{3}}{2} - \sin^{-1} \frac{\sqrt{3}-1}{2} \right] \\ &= -\sin^{-1} \frac{1-\sqrt{3}}{2} + \sin^{-1} \frac{\sqrt{3}-1}{2} \\ &= 2 \sin^{-1} \left[\frac{\sqrt{3}-1}{2} \right] \quad [\because \sin^{-1}(-x) = -\sin^{-1} x] \end{aligned}$$

उत्तर

प्रश्न 29. $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

हल : $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$\begin{aligned} &= \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} \times \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} dx \\ &= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx = \int_0^1 (\sqrt{1+x}+\sqrt{x}) dx \\ &= \frac{2}{3} \left[(1+x)^{3/2} + x^{3/2} \right]_0^1 = \frac{2}{3} [2^{3/2} + 1 - 1] \\ &= \frac{2}{3} \cdot 2\sqrt{2} = \frac{4\sqrt{2}}{3} \end{aligned}$$

उत्तर

प्रश्न 30. $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x} dx$

हल : $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x} dx$

मान लीजिए $\sin x - \cos x = t$ हो, तब $(\cos x + \sin x) dx = dt$

जब $x = \frac{\pi}{4}$ तो $t = 0$ तथा जब $x = 0$ तो $t = -1$

$(\sin x - \cos x)^2 = 1 - 2 \sin x \cos x = 1 - \sin 2x$

$\therefore \sin 2x = 1 - (\sin x - \cos x)^2 = 1 - t^2$

$$= \int_{-1}^0 \frac{dt}{9+16(1-t^2)} = \int_{-1}^0 \frac{dt}{25-16t^2}$$

$$\begin{aligned}
&= \frac{1}{16} \int_{-1}^0 \frac{dt}{\frac{25}{16} - t^2} \\
&= \frac{1}{16} \cdot \frac{1}{2} \cdot \frac{4}{5} \left[\log \left(\frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right) \right]_{-1}^0 \\
&= \frac{1}{40} [(\log 1 - (\log 1 - \log 9))] \\
&= \frac{1}{40} \left(\log \frac{1}{9} \right) = \frac{1}{40} \log 9.
\end{aligned}$$

उत्तर

प्रश्न 31. $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x) dx$

हल : $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x) dx = 2 \int_0^{\pi/2} \sin x \cos x \tan^{-1}(\sin x) dx$

मान लीजिए $\sin x = t$ हो, तब, $\cos x dx = dt$

जब $x = \frac{\pi}{2}$ तो $t = 1$, तथा जब $x = 0$ तो $t = 0$

$$= 2 \int_0^1 t \tan^{-1} t dt$$

खण्डशः समाकलन करने पर,

$$\begin{aligned}
&= 2 \left[\tan^{-1} t \cdot \frac{t^2}{2} \right]_0^1 - 2 \int_0^1 \frac{1}{1+t^2} \times \frac{t^2}{2} dt \\
&= \frac{\pi}{4} - \int_0^1 \frac{1+t^2-1}{1+t^2} dt = \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+t^2} \right) dt \\
&= \frac{\pi}{4} - [t - \tan^{-1} t]_0^1 = \frac{\pi}{4} - \frac{1}{2} (1 - \tan^{-1} 1 - 0) \\
&= \frac{\pi}{4} - \left(1 - \frac{\pi}{4} \right) = \frac{\pi}{2} - 1.
\end{aligned}$$

उत्तर

प्रश्न 32. $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx.$

हल :

$$I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \quad \dots(i)$$

$\therefore$

$$\begin{aligned}
I &= \int_0^{\pi} \frac{(\pi - x) \tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx \\
&\quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{\pi} \frac{-(\pi-x) \tan x}{\sec x - \tan x} dx \\
&\quad [\because \sec(\pi-x) = -\sec x \text{ एवं } \tan(\pi-x) = -\tan x] \\
I &= \int_0^{\pi} \frac{(\pi-x) \tan x}{\sec x + \tan x} dx \quad \dots(ii)
\end{aligned}$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$\begin{aligned}
2I &= \int_0^{\pi} \frac{[x + (\pi-x)] \tan x}{\sec x + \tan x} dx \\
&= \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx \\
&= \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx \\
&= \pi \int_0^{\pi} \frac{\sin x dx}{1 + \sin x} \\
&= \pi \int_0^{\pi} \frac{1 + \sin x - 1}{1 + \sin x} dx = \pi \int_0^{\pi} \left(1 - \frac{1}{1 + \sin x}\right) dx \\
&= \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x} dx \\
&= \pi^2 - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx \\
&= \pi^2 - \pi \int_0^{\pi} (\sec^2 x - \sec x \tan x) dx \\
&= \pi^2 - \pi [\tan x - \sec x]_0^{\pi} \\
&= \pi^2 - \pi [(\tan \pi - \sec \pi) - \tan 0 - \sec 0] \\
&= \pi^2 - \pi [1 + 1] = \pi^2 - 2\pi \\
\therefore I &= \frac{\pi^2 - 2\pi}{2} = \frac{\pi}{2} [\pi - 2].
\end{aligned}$$

उत्तर

प्रश्न 33. $\int_1^4 (|x-1| + |x-2| + |x-3|) dx$.

हल : मान लीजिए

$$\begin{aligned}
I &= \int_1^4 (|x-1| + |x-2| + |x-3|) dx \\
&= \int_1^4 |x-1| dx + \int_1^4 |x-2| dx + \int_1^4 |x-3| dx \\
&= I_1 + I_2 + I_3 \text{ (माना)} \\
I_1 &= \int_1^4 |x-1| dx = \int_1^4 (x-1) dx \quad [\because x > 1] \\
&= \left[\frac{x^2}{2} - x \right]_1^4 = (8-4) - \left(\frac{1}{2} - 1 \right) = 4 + \frac{1}{2} = 4\frac{1}{2} \quad \dots(i)
\end{aligned}$$

$$I_2 = \int_1^4 |x-2| dx = \int_1^2 |x-2| dx + \int_2^4 |x-2| dx$$

$$= -\int_1^2 (x-2) dx + \int_2^4 (x-2) dx$$

∴ जब $1 < x < 2$ तो $|x-2| = -(x-2)$

तथा जब $2 < x < 4$ तो $|x-2| = x-2$

$$\therefore I_2 = \left[-\frac{x^2}{2} + 2x \right]_1^2 + \left[\frac{x^2}{2} - 2x \right]_2^4$$

$$= \left[\left(-\frac{4}{2} + 4 \right) - \left(-\frac{1}{2} + 2 \right) \right] + \left[\left(\frac{16}{2} - 8 \right) - \left(\frac{4}{2} - 4 \right) \right]$$

$$= \left[2 - \frac{3}{2} + 2 \right] = 4 - \frac{3}{2} = \frac{5}{2} = 2\frac{1}{2} \quad \dots(ii)$$

और

$$I_3 = \int_1^4 |x-3| dx = \int_1^3 |x-3| dx + \int_3^4 |x-3| dx$$

∴ जब $1 < x < 3$ तो $|x-3| = -(x-3)$

तथा जब $3 < x < 4$ तो $|x-3| = x-3$

$$\therefore I_3 = \int_1^3 -(x-3) dx + \int_3^4 (x-3) dx$$

$$= \left[-\frac{x^2}{2} + 3x \right]_1^3 + \left[\frac{x^2}{2} - 3x \right]_3^4$$

$$= \left[\left(-\frac{9}{2} + 9 \right) - \left(-\frac{1}{2} + 3 \right) \right] + \left[\left(\frac{16}{2} - 12 \right) - \left(\frac{9}{2} - 9 \right) \right]$$

$$= \left(\frac{9}{2} - \frac{5}{2} \right) + \left(-4 + \frac{9}{2} \right) = \frac{5}{2} \quad \dots(iii)$$

समीकरण (i), (ii) व (iii) को जोड़ने पर,

$$I_1 + I_2 + I_3$$

$$= \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

उत्तर

निम्नलिखित को सिद्ध कीजिए (प्रश्न 34 से 39 तक) :

प्रश्न 34. $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$

हल : बायाँ पक्ष : $\int_1^3 \frac{dx}{x^2(x+1)}$

यहाँ

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\therefore 1 = Ax(x+1) + B(x+1) + Cx^2$$

$$x=0 \text{ लेने पर, } 1 = B \times 1 \text{ या } B = 1$$

$$x=-1 \text{ लेने पर, } 1 = C(-1)^2 \text{ या } C = 1$$

x^2 के गुणांकों की तुलना करने पर

$$0 = A + C \text{ या } A = -C = -1$$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$\begin{aligned} \therefore \int_1^3 \frac{1}{x^2(x+1)} dx &= \int_1^3 -\frac{1}{x} dx + \int_1^3 \frac{1}{x^2} dx + \int_1^3 \frac{1}{x+1} dx \\ &= [-\log |x|]_1^3 - \left[\frac{1}{x} \right]_1^3 + [\log |x+1|]_1^3 \\ &= (-\log 3 + \log 1) + \left(-\frac{1}{3} + \frac{1}{1} \right) + \log 4 - \log 2 \\ &= -\log 3 + \frac{2}{3} + 2 \log 2 - \log 2 \\ &= \frac{2}{3} + \log \frac{2}{3} = \text{दायाँ पक्ष।} \end{aligned}$$

प्रश्न 35. $\int_0^1 x e^x dx = 1.$

हल : बायाँ पक्ष $\int_0^1 x e^x dx$

x को पहला फलन मानकर खण्डशः समाकलन करने पर

$$\begin{aligned} &= [x e^x]_0^1 - \int_0^1 1 \cdot e^x dx \\ &= 1 \cdot e^1 - [e^x]_0^1 = e - (e - 1) = 1 = \text{दायाँ पक्ष।} \end{aligned}$$

प्रश्न 36. $\int_{-1}^1 x^{17} \cos^4 x dx = 0.$

हल : मान लीजिए

$$\begin{aligned} I &= \int_{-1}^1 x^{17} \cos^4 x dx \\ f(x) &= x^{17} \cos^4 x, f(-x) = (-x)^{17} \cos^4(-x) dx \\ &= -x^{17} \cos^4 x \end{aligned}$$

$$\therefore f(-x) = -f(x) \int_{-a}^a f(x) dx = 0 \text{ यदि } f \text{ विषम फलन हो}$$

$$\therefore I = 0 \text{ अर्थात् } \int_{-1}^1 x^{17} \cos^4 x dx = 0. \quad \text{इति सिद्धम्।}$$

प्रश्न 37. $\int_0^{\pi/2} \sin^3 x dx = \frac{2}{3}.$

हल : मान लीजिए

$$\begin{aligned} I &= \int_0^{\pi/2} \sin^3 x dx \\ \therefore \sin 3x &= 3 \sin x - 4 \sin^3 x \end{aligned}$$

$$\therefore \sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$\therefore I = \frac{1}{4} \int_0^{\pi/2} (3 \sin x - \sin 3x) dx$$

$$\begin{aligned}
&= \frac{1}{4} \left[-3 \cos x + \frac{\cos 3x}{3} \right]_0^{\pi/2} \\
&= \frac{1}{4} \left[0 - \left(-3 + \frac{1}{3} \right) \right] = \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}.
\end{aligned}$$

इति सिद्धम्।

प्रश्न 38. $\int_0^{\pi/4} 2 \tan^3 x \, dx = 1 - \log 2$.

हल : बायाँ पक्ष: $\int_0^{\pi/4} 2 \tan^3 x \, dx = 2 \int_0^{\pi/4} \tan^3 x \, dx$

$$\begin{aligned}
&= 2 \int_0^{\pi/4} \tan x \cdot \tan^2 x \, dx \\
&= 2 \int_0^{\pi/4} \tan x (\sec^2 x - 1) \, dx \\
&= 2 \int_0^{\pi/4} \tan x \sec^2 x \, dx - 2 \int_0^{\pi/4} \tan x \, dx = 2[I_1 - I_2]
\end{aligned}$$

यहाँ $\tan x = t$ तब $\sec^2 x \, dx = dt$

जब $x = 0$ हो तब $t = 0$ और जब $x = \frac{\pi}{4}$ हो, तब $t = 1$

$\therefore I_1 = \int_0^1 dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2}$

और $I_2 = \int_0^{\pi/4} \tan x \, dx = -[\log \cos x]_0^{\pi/4}$

$$\begin{aligned}
&= -\left[\log \cos \frac{\pi}{4} - \log \cos 0 \right] \\
&= -\left[\log \frac{1}{\sqrt{2}} - \log 1 \right] = -\log \frac{1}{\sqrt{2}} = \frac{1}{2} \log 2
\end{aligned}$$

अब $2(I_1 - I_2) = 2 \left[\frac{1}{2} - \frac{1}{2} \log 2 \right] = 1 - \log 2 = \text{दायाँ पक्ष।}$

प्रश्न 39. $\int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - 1$.

हल : बायाँ पक्ष : $\int_0^1 \sin^{-1} x \cdot 1 \, dx$

$\sin^{-1} x$ को पहला फलन मानकर खण्डशः समाकलन करने पर

$$\begin{aligned}
&= \left[(\sin^{-1} x) x \right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot x \, dx \\
&= \left[\sin^{-1} x \right]_0^1 + \frac{1}{2} \int_0^1 \frac{-2x}{\sqrt{1-x^2}} \, dx \\
&= \frac{\pi}{2} + \frac{1}{2} \times 2 \left[\sqrt{1-x^2} \right]_0^1 \\
&= \frac{\pi}{2} - 1 = \text{दायाँ पक्ष।}
\end{aligned}$$

प्रश्न 40. योगफल की सीमा के रूप में $\int_0^1 e^{2-3x} dx$ का मान ज्ञात कीजिए।

हल : मान लीजिए $I = \int_0^1 e^{2-3x} dx$

$$\therefore \int_a^b f(x) dx = \lim_{h \rightarrow 0} h[f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]$$

जबकि $h = \frac{1-0}{n} = \frac{1}{n}$ या $nh = 1$ [$\because a = 0, b = 1$]

$$f(x) = e^{2-3x}, f(a) = e^{2-0} = e^2, f(a+h) = e^{2-3h}$$

$$f(a+2h) = e^{2-6h}, f(a+3h) = e^{2-9h}, \dots$$

$$f(a+(n-1)h) = e^{2-3(n-1)h}$$

$$\therefore \int_0^1 e^{2-3x} dx = \lim_{h \rightarrow 0} h[e^2 + e^{2-3h} + e^{2-6h} + e^{2-9h} + \dots + e^{2-3(n-1)h}]$$

जबकि $nh \rightarrow b-a = 1-0 = 1$

$$\therefore I = \lim_{h \rightarrow 0} h e^2 [1 + e^{-3h} + e^{-6h} + e^{-9h} + \dots + e^{-3(n-1)h}]$$

$$= \lim_{h \rightarrow 0} h e^2 \left[\frac{e^{-3nh} - 1}{e^{-3h} - 1} \right] \left[\because S = \frac{a(r^n - 1)}{a - 1} \text{ गुणोत्तर श्रेणी के लिए} \right]$$

$$= \lim_{h \rightarrow 0} h e^2 (e^{-3nh} - 1) \left(\frac{-3h}{e^{-3h} - 1} \right) \times \frac{1}{-3h}$$

अब $nh \rightarrow 1, \lim_{h \rightarrow 0} \frac{-3h}{e^{-3h} - 1} = \log e = 1$

$$\therefore I = e^2 (e^3 - 1) \cdot 1 \left(-\frac{1}{3} \right)$$

$$= -\frac{1}{3} [e^{-1} - e^2] = \frac{1}{3} \left(e^2 - \frac{1}{e} \right).$$

उत्तर

प्रश्न 41 से 44 तक के प्रश्नों में सही उत्तर का चयन कीजिए :

प्रश्न 41. $\int \frac{dx}{e^x + e^{-x}}$ बराबर है :

(A) $\tan^{-1}(e^x) + C$

(B) $\tan^{-1}(e^{-x}) + C$

(C) $\log(e^x - e^{-x}) + C$

(D) $\log(e^x + e^{-x}) + C$

हल : $\int \frac{dx}{e^x + e^{-x}} = \int \frac{dx}{e^x + \frac{1}{e^x}} = \int \frac{e^x dx}{e^{2x} + 1}$

मान लीजिए $e^x = t$ हो, तब $e^x dx = dt$

$$= \int \frac{dt}{t^2 + 1} = \tan^{-1} t + C = \tan^{-1} e^x + C.$$

अतः विकल्प (A) सही है।

उत्तर

प्रश्न 42. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ बराबर है :

- (A) $\frac{-1}{\sin x + \cos x} + C$ (B) $\log |\sin x + \cos x| + C$
 (C) $\log |\sin x - \cos x| + C$ (D) $\frac{1}{(\sin x + \cos x)^2}$

हल :
$$\begin{aligned} \int \frac{\cos 2x}{(\sin x + \cos x)^2} dx &= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx \\ &= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx \\ &= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx \end{aligned}$$

मान लीजिए $\cos x + \sin x = t$ हो, तब $-\sin x + \cos x dx = dt$

$$= \int \frac{dt}{t} = \log |t| + C = \log |\sin x + \cos x| + C$$

अतः विकल्प (B) सही है।

उत्तर

प्रश्न 43. यदि $f(a+b-x) = f(x)$, तो $\int_a^b xf(x) dx$ बराबर है :

- (A) $\frac{a+b}{2} \int_a^b f(b-x) dx$ (B) $\frac{a+b}{2} \int_a^b f(b+x) dx$
 (C) $\frac{b-a}{2} \int_a^b f(x) dx$ (D) $\frac{a+b}{2} \int_a^b f(x) dx$

हल : मान लीजिए

$$\begin{aligned} I &= \int_a^b xf(x) dx \\ &= \int_a^b (a+b-x)f(a+b-x) dx \\ &\quad \left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right] \\ &= \int_a^b f(a+b-x)f(x) dx \\ \therefore I &= \int_a^b [(a+b)f(x) - xf(x)] dx \\ &= (a+b) \int_a^b f(x) dx - \int_a^b xf(x) dx \\ &= (a+b) \int_a^b f(x) dx - I \\ 2I &= (a+b) \int_a^b f(x) dx \\ \therefore I &= \frac{a+b}{2} \int_a^b f(x) dx \end{aligned}$$

अतः विकल्प (D) सही है।

उत्तर

प्रश्न 44. $\int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx$ का मान है :

(A) 1

(B) 0

(C) -1

(D) $\frac{\pi}{4}$

हल : मान लीजिए

$$I = \int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx = \int_0^1 \tan^{-1} \left[\frac{x+(x-1)}{1-x(x-1)} \right] dx$$

$$= \int_0^1 [\tan^{-1} x + \tan^{-1} (x-1)] dx \quad \dots(i)$$

$$= \int_0^1 [\tan^{-1} (1-x) + \tan^{-1} (1-x-1)] dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$= \int_0^1 [-\tan^{-1} (x-1) - \tan^{-1} x] dx$$

$$I = -\int_0^1 \tan^{-1} x + \tan^{-1} (x-1) dx \quad \dots(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = 0 \text{ या } I = 0$$

अतः विकल्प (B) सही है।

उत्तर